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About И-quantifiers

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Abstract. Gabbay and Pitts observed that the Fraenkel–Mostowski model of set-theory supports useful notions of "name-abstraction" and "fresh-name". In order to understand their work in a more general setting we introduce the notions of И-units and И-relations in a regular category **D**. A И-relation is given by a functor $\mathbb{A} \# (\_) : \mathbf{D} \longrightarrow \mathbf{D}$ and we show that in the case that **D** is a topos then $\mathbb{A} \# (\_)$ has a right adjoint $[\mathbb{A}](\_)$ that can be thought of as an object of abstractions. We also explore the existence of a right adjoint to $[\mathbb{A}](\_)$ and relate it to the "name swapping" operations considered as fundamental by Gabbay and Pitts. We present many examples of categories where this notions occur and we relate the results here with Pitts' Nominal Logic.

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1. Introduction

Gabbay and Pitts have convincingly argued in [8] that the Fraenkel–Mostowski permutation model of set theory is a convenient setting for developing the meta-mathematics of formal systems involving variable binding operations. They suggest to think of the sets in this model as sets of (finite) terms with variables in a distinguished set of names $\mathbb{A}$. Every element has an associated finite set of names called its *support* which is to be thought of as the set of names that appear free in the element. It is then possible to define for any X, $\mathbb{A} \# X = \{(a, x) \mid a \text{ is not in the support of } x\}$ and write $a \# x$ if $(a, x) \in \mathbb{A} \# X$. A first key property of this notion of support is that, in any context, we can always choose a fresh name. In other words, the following sequent holds.

$$x \in X \vdash (\exists a \in \mathbb{A})(a \# x). \qquad \text{(Fresh)}$$

As explained in [8] many situations involving fresh names have the following form: first we choose *some* fresh name with a particular property, but later we need the fact that *any* such name will do. Consider a subset U of $\mathbb{A} \# X$. An element of U is a pair (a, x) such that a is fresh. Think of this as a fresh name a (in a context X)

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with a particular property U. A second key property is that for any such X and U the following sequent holds in the Fraenkel–Mostowski model.

$$x \in X \vdash (\exists a \in \mathbb{A})(a \# x \wedge (a, x) \in U) \rightarrow (\forall b \in \mathbb{A})(b \# x \rightarrow (b, x) \in U).$$

So we can always choose a fresh name by (Fresh) and then use the sequent above to conclude that it does not matter which one. It is easy to show (see also Lemma 3.1) that the conjunction of the two sequents above is equivalent to the validity of the following sequent.

$$x \in X \vdash (\forall b \in \mathbb{A})(b \# x \rightarrow (b, x) \in U) \leftrightarrow (\exists a \in \mathbb{A})(a \# x \wedge (a, x) \in U). \qquad (\dagger)$$

Gabbay and Pitts suggest to write $(\text{И}a)((a, x) \in U)$ for any of the two equivalent formulas and to think of И as a quantifier expressing "for some/any new (fresh) name". Using this novel quantifier and certain "name swapping" actions $\mathbb{A} \times \mathbb{A} \times X \longrightarrow X$ they define an operation $[\mathbb{A}](\_)$ of "atom-abstraction" and show that it has very good properties. In particular, that it can be used in combination with products and coproducts to form inductively defined sets that represent syntax modulo α-conversion.

The purpose of this paper is to give an account of this new quantifier in the same spirit that the usual quantifiers $\forall$ and $\exists$ are understood as adjoints to substitution [13, 17]. This perspective will expose the constructions in [8] as adjunctions and many of their properties as consequences of this fact. The basic properties of И-quantifiers can be formulated in a regular category, but the main results will require an underlying topos. (This does not seem to be a strong restriction since all the models we are aware of are toposes.) Because of this the reader will find the material here easier to follow if he is more or less familiar with the first five chapters of [14], with regular categories [1, 3], with toposes (Chapters I, IV, V and VI of [15]) and with the definition of a monoidal category [5]. On the other hand, we would like to stress that only elementary aspects of these areas are used and that we have included many details in the proofs. In this way, anyone keen on adjunctions and familiar with [8] should be able to understand the results in our paper. In particular, we hope that people working on mathematics of formal systems with variable binding operations will find the material interesting even if they are only vaguely acquainted with toposes.

In order to motivate the results in this paper it is useful to introduce a somewhat minimal notion of an *object of names*. Let $\mathbf{D}$ be a monoidal category with tensor $\# : \mathbf{D} \times \mathbf{D} \longrightarrow \mathbf{D}$ and unit I. Assume also that $\mathbf{D}$ has finite products and finite coproducts. Let $\mathbb{A}$ be an object of $\mathbf{D}$ such that the functor $\mathbb{A} \# (\_) : \mathbf{D} \longrightarrow \mathbf{D}$ has a right adjoint $[\mathbb{A}](\_)$ which itself has a further right adjoint $(\_)_{[\mathbb{A}]}$ so that $\mathbb{A} \# (\_) \dashv [\mathbb{A}](\_) \dashv (\_)_{[\mathbb{A}]} : \mathbf{D} \longrightarrow \mathbf{D}$. Denote by $\iota : I \longrightarrow [\mathbb{A}]\mathbb{A}$ the transposition of $\mathbb{A} \# I \cong \mathbb{A} \longrightarrow \mathbb{A}$ given by the identity. Suppose also that there is a transformation $\rho : \mathbb{A} \# X \longrightarrow X$ (natural in X) and denote by $\tau_{\mathbb{A}} : \mathbb{A} \longrightarrow [\mathbb{A}]\mathbb{A}$ the transposition of $\rho_{\mathbb{A}} : \mathbb{A} \# \mathbb{A} \longrightarrow \mathbb{A}$. Finally, assume that the map $[\iota, \tau] : I + \mathbb{A} \longrightarrow [\mathbb{A}]\mathbb{A}$ is an isomorphism. For the purpose of this introduction let us say that the data above presents $\mathbb{A}$ as an *object of names*. (Actually, the monoidal structure is not that relevant. We mention it here to ease the presentation, the technical results will only depend on the functor $\mathbb{A}\ \#\ (\_)$ and the object of names will appear as $\mathbb{A}\ \#\ 1$.)

The idea is again to think of the objects of **D** as sets of terms with variables coming from the object of names $\mathbb{A}$. The tensor provides some way of "pairing" terms and ρ is some sort of "projection". The object $[\mathbb{A}]X$ should be thought of as a set of "abstractions" $(\lambda a.x)$ with $a \in \mathbb{A}$ and $x \in X$. The map ι is pointing to the "term" $(\lambda a.a)$ and $\tau_{\mathbb{A}}$ takes a name b and builds the "term" $(\lambda a.b)$ with λa not binding b. At present we are unable to give an intuitive reading of the rightmost adjoint $(\_)_{[\mathbb{A}]}$ but notice that it implies that $[\mathbb{A}](\_)$ preserves colimits.

For example, consider the topos $\mathcal{F}$ studied in [7]. This is the topos of functors from the essentially small category of finite sets and functions to the category of sets. In this case, take the monoidal structure # to be given by finite products. If we take $\mathbb{A}$ to be the inclusion of finite sets into sets and ρ to be the natural projection given by finite products then we obtain an object of names in the sense above. For details see [7] where $\mathbb{A}$ is denoted by V and $[\mathbb{A}](\_) : \mathcal{F} \longrightarrow \mathcal{F}$ is denoted by δ. The fact that $[\mathbb{A}](\_)$ has both a left and a right adjoint implies that it preserves both limits and colimits and this is used to build initial algebras for the so called *binding signatures*. In particular the presheaf of λ-terms modulo α-equivalence is shown to be isomorphic to the initial algebra for the functor $[\mathbb{A}](\_) + (\_) \times (\_) + \mathbb{A} : \mathcal{F} \longrightarrow \mathcal{F}$. Other examples of objects of names can be found in [9]. Moreover, some of the results in [8] can be seen as showing that certain "swapping" operations $\mathbb{A} \times \mathbb{A} \times X \longrightarrow X$ present in the Schanuel topos **Sch** (see Corollary III.9.3 in [15] and Section 3 below) imply that $\mathbb{A}$ is an object names.

We can now briefly outline the contents of this paper as follows. First we introduce an abstract formulation of И-quantifiers and then we show how this definition can be used to obtain an object of names in the sense above. In contrast with [8], we will try to make no reference to "swapping" operations until it is absolutely necessary. A referee suggested that we compare the present work with Pitts' Nominal Logic so we do this in Section 6. Finally, in Section 7, we build new examples of И-quantifiers. Let us now introduce the main definitions and notation in order to describe the contents of the paper in more detail. Let **D** be a regular category and let $F : \mathbf{D} \longrightarrow \mathbf{D}$ be an endofunctor.

DEFINITION 1.1. A И-*unit* for F is a natural transformation $\theta : F \longrightarrow Id$ such that for every X, $\theta_X^* : Sub(X) \longrightarrow Sub(FX)$ is an isomorphism of posets.

This is equivalent to require, for every X, the existence of a right adjoint $\forall_{\theta_X}$ to θ_X^* such that $\exists_{\theta_X} = \forall_{\theta_X}$ (see Section 2). We can then write И for any/both adjoints $\exists_{\theta_X}$ and $\forall_{\theta_X}$. In Section 2 we prove the basic consequences of the existence of a И-unit. The notion of И-unit clearly admits several generalizations. For example, by allowing $\theta : F \longrightarrow G$ or by considering transformations more general than θ^*. Instead of exploring these we are going to restrict the definition in order to bring it closer to what happens in the Schanuel topos **Sch**. Intuitively, we require that FX should be a (well-behaved) "property" of pairs (name, context). (In order to avoid confusion let us mention the following convention: a functor that *preserves pullbacks* does not necessarily preserve the terminal object.)

DEFINITION 1.2. A И-*relation* is a pullback-preserving $F : \mathbf{D} \longrightarrow \mathbf{D}$ together with a И-unit $\theta : F \longrightarrow Id$ such that the map $\langle F!_X, \theta_X \rangle : FX \longrightarrow F1 \times X$ is mono for every X.

For any И-unit let us denote $F1$ by $\mathbb{A}$. Notice that the definition of И-relation makes F a subfunctor of $\mathbb{A} \times (\_)$. Because of this, we write $\mathbb{A} \# (\_)$ instead of F. We then have that $\mathbb{A} \# 1 = \mathbb{A}$ and that $\langle F!, \theta_1 \rangle : \mathbb{A} \longrightarrow \mathbb{A} \times 1 \cong id_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}$. The intuition should be that in a topos $\mathbf{D}$ with a И-relation $\mathbb{A} \# (\_)$ the objects can be thought of as sets of terms with variables coming from the set of names $\mathbb{A}$. Then, $\mathbb{A} \# X$ can be thought of as the set whose elements are those pairs $(a, x) \in \mathbb{A} \times X$ such that a does not appear free in x. In Section 3 we introduce some notation to deal with И-relations, introduce the Schanuel topos and prove that it has a И-relation. In Section 4 we prove the result below (without any reference to "name swapping" operations).

PROPOSITION 1.3. *Let* $\mathbf{D}$ *be a topos with a* И-*relation* $\mathbb{A} \# (\_) : \mathbf{D} \longrightarrow \mathbf{D}$. *Then* $\mathbb{A} \# (\_)$ *has a right adjoint that we denote by* $[\mathbb{A}](\_) : \mathbf{D} \longrightarrow \mathbf{D}$.

In a brief subsection we study the assumption that $\mathbb{A} \# \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ is the complement of the diagonal. We show that in this case the object $[\mathbb{A}]\mathbb{A}$ is canonically iso to $1 + \mathbb{A}$ and that the formula presented in [8] to calculate the support of an element in $[\mathbb{A}]X$ holds in the present more general context.

Let us now introduce some notation in order to discuss further results. For a И-relation $\mathbb{A} \# (\_)$ we write $\zeta_X : \mathbb{A} \# X \rightarrowtail \mathbb{A} \times X$ for the map $\langle F!_X, \theta_X \rangle$. For a generalized element $(a, x) \in \mathbb{A} \times X$ we write $a \# x$ for $(a, x) \in \mathbb{A} \# X$, that is, $a \# x$ if and only if (a, x) factors through $\zeta_X : \mathbb{A} \# X \longrightarrow \mathbb{A} \times X$. Let $\epsilon : \mathbb{A} \# [\mathbb{A}]X \longrightarrow X$ be the counit of the adjunction stated in Proposition 1.3. For any $(a, f) \in \mathbb{A} \# [\mathbb{A}]X$ we write fa instead of $\epsilon(a, f)$.

After Proposition 1.3 (and with our minimal notion of an object of names in mind) the natural step forward would be to show that under the hypotheses we have made then there exists a further right adjoint $(\_)_{[\mathbb{A}]}$. Unfortunately we are unable to do this. Instead, we show in Section 5 that the existence of $(\_)_{[\mathbb{A}]}$ is implied by the assumption that $[\mathbb{A}]X$ is a "good quotient" of $\mathbb{A} \times X$. By this, we mean a quotient that allows us to define maps from $[\mathbb{A}]X$ easily in terms of the representatives in $\mathbb{A} \times X$ (see Lemma 6.3 in [8]).

DEFINITION 1.4. We say that $\mathbf{D}$ has *pre-binders* if there is a transformation $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ (natural in X) such that for every map $f : \mathbb{A} \times X \longrightarrow Y$, there exists a unique map $g : [\mathbb{A}]X \longrightarrow Y$ such that $g.\lambda = f$ if and only if the sequent $a \in \mathbb{A}, x \in X \vdash a \# f(a, x)$ holds.

Let $HX = \{f \in X^{\mathbb{A}} \mid (\forall a \in \mathbb{A})(a \mathbin{\#} fa)\}$. This defines an endofunctor H on $\mathbf{D}$ and the perspective we want to stress is the following.

LEMMA 1.5. *Let* $\mathbf{D}$ *be a topos with a* И*-relation* $\mathbb{A} \mathbin{\#} (\_) : \mathbf{D} \longrightarrow \mathbf{D}$. *Then the functor* $H : \mathbf{D} \longrightarrow \mathbf{D}$ *is right adjoint to* $[\mathbb{A}](\_)$ *if and only if* $\mathbf{D}$ *has pre-binders.*

The existence of a good quotient $\mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ seems to be quite useful in applications (this being already clear from their use in [8]) so we will study them in a little more detail. For this it is useful to introduce the following notion.

DEFINITION 1.6. A *binder* for an object X is a map $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ such that there exists a map $\lambda' : \mathbb{A} \times X \longrightarrow \mathbb{A} \mathbin{\#} [\mathbb{A}]X$ making the following diagram commutes.

$$\begin{tikzcd}
 & \mathbb{A} \times X \arrow[dl, "{\langle \pi_0, \lambda \rangle}"'] \arrow[d, "\lambda'"] \arrow[dr, "\pi_1"] & \\
\mathbb{A} \times [\mathbb{A}]X & \mathbb{A} \mathbin{\#} [\mathbb{A}]X \arrow[l, tail, "\zeta"] \arrow[r, "\epsilon"'] & X
\end{tikzcd}$$

If we write $(\lambda a.x)$ for λ applied to a (generalized) element $(a, x) \in \mathbb{A} \times X$ then λ is a binder if and only if the following hold in the internal logic.

(1) $a \in \mathbb{A}, x \in X \vdash a \mathbin{\#} (\lambda a.x)$,
(2) $a \in \mathbb{A}, x \in X \vdash (\lambda a.x)a = x$.

We say that a topos with a И-relation *has binders* if there is a binder for every object. The basic properties of binders are studied also in Section 5 where in particular we show that the existence of binders implies the existence of pre-binders (we do not know if the converse holds) and we show that under a mild condition (valid in **Sch**) the existence of binders is equivalent to the existence of "name swapping operations" $\mathbb{A} \mathbin{\#} (\mathbb{A} \times X) \longrightarrow X$. These swapping operations are the fundamental notion in [8]. They are also fundamental in Pitts' more recent [18] so we briefly sketch in Section 6 what is the relation of Pitts' work with the one presented here. It is fair to ask at this point if the existence of a rightmost adjoint $(\_)_{[\mathbb{A}]}$ implies the existence of (pre-)binders. The answer is no. In Section 7 we build new examples of toposes with И-quantifiers and in particular show that while the strings of adjoints $\mathbb{A} \mathbin{\#} (\_) \dashv [\mathbb{A}](\_) \dashv (\_)_{[\mathbb{A}]}$ survive slicing, pre-binders do not. It should also be noted that at no point in the paper the underlying topos is assumed to be boolean and in Section 7 we also build examples of non-Boolean toposes with И-quantifiers. Section 8 is devoted to the conclusions.

2. И-units

In this section we prove the basic properties of И-units (Definition 1.1) after a brief recap on regular categories. We say that a category is *regular* if it has finite limits and stable regular-epi/mono factorizations [1]. Stability induces, for every map $f : Y \longrightarrow X$, a monotone $\exists_f : \mathit{Sub}(Y) \longrightarrow \mathit{Sub}(X)$ between the posets of subobjects that is left adjoint to the monotone $f^* : \mathit{Sub}(X) \longrightarrow \mathit{Sub}(Y)$ given by pulling back along f. A right adjoint to f^* need not always exist but when it does, it is denoted by $\forall_f$. For example, these right adjoints exist in locally Cartesian closed categories. We will assume familiarity with these ideas, but for reasons that will become clear, let us emphasize the following simple observations (whose proofs are omitted).

LEMMA 2.1. *For any $f : Y \longrightarrow X$ in a regular category* $\mathbf{D}$*, the following are equivalent.*

(1) $\exists_f . f^* = \mathit{id} : \mathit{Sub}(X) \longrightarrow \mathit{Sub}(X)$,
(2) *f is a regular epi,*
(3) *f^* is mono.*

Moreover, if $\forall_f$ exists then the above are also equivalent to the following.

(4) $\forall_f \leq \exists_f$.

On the other hand, we have the following.

LEMMA 2.2. *For any $f : Y \longrightarrow X$ in a regular category* $\mathbf{D}$*, the following are equivalent.*

(1) $f^* . \exists_f = \mathit{id} : \mathit{Sub}(Y) \longrightarrow \mathit{Sub}(Y)$,
(2) *$f^* : \mathit{Sub}(X) \longrightarrow \mathit{Sub}(Y)$ is epi.*

Moreover, if $\forall_f$ exists then the above are also equivalent to the following.

(3) $\exists_f \leq \forall_f$.

So let us conclude that, f^* is an isomorphism if and only if f^* has a right adjoint $\forall_f$ and moreover, $\exists_f = \forall_f$. This explains the connection between Definition 1.1 and the formula (†) presented in the beginning of the introduction (see also Lemma 3.1). For the remaining of the section let $\mathbf{D}$ be a regular category and let $F : \mathbf{D} \longrightarrow \mathbf{D}$ be an endofunctor. Recall (Definition 1.1) that a И-unit for F is a natural $\theta : F \longrightarrow \mathit{Id}$ such that θ^* is an iso. By Lemmas 2.1 and 2.2 this is equivalent to the fact that for every X, $\theta_X^* : \mathit{Sub}(FX) \longrightarrow \mathit{Sub}(X)$ has both a left adjoint $\exists$ and a right adjoint $\forall$ and moreover $\exists = \forall$. Denote by И any of the two equal adjoints $\exists = \forall$. Think of this as a new quantifier $\mathit{Sub}(FX) \longrightarrow \mathit{Sub}(X)$ for which the usual properties $\exists_\theta \dashv \theta^* \dashv \forall_\theta$ have merged into $\theta^* \dashv \text{И} \dashv \theta^*$.

LEMMA 2.3. *If F has a И-unit then the following hold.*

(1) *For every X, $\theta_X : FX \longrightarrow X$ is a regular epi.*
(2) *F is faithful.*
(3) *F preserves regular epis.*

Proof. Item (1) follows from Lemma 2.1 and item (2) follows from item (1) using naturality of θ. To prove (3) let $q : X \twoheadrightarrow Q$ be a regular epi. Then $q^* : Sub(Q) \longrightarrow Sub(X)$ is mono and we can consider the following diagram.

$$\begin{tikzcd} Sub(Q) \arrow[r, "q^*"] \arrow[d, "\theta_Q^*"', "\cong"] & Sub(X) \arrow[d, "\theta_X^*"', "\cong"] \\ Sub(FQ) \arrow[r, "(Fq)^*"'] & Sub(FX) \end{tikzcd}$$

Then $(Fq)^*$ is also mono and so Fq is a regular epi (Lemma 2.1). $\square$

As a subfunctor of a product, the underlying functor of a И-relation (Definition 1.2) has to preserve monos. This last property has a number of useful consequences. Before we state them we introduce a useful definition.

DEFINITION 2.4. Let $\alpha : G \longrightarrow H$ be a natural transformation between functors G and H. A map $f : Y \longrightarrow X$ is called *α-stable* if the following commutative square is a pullback.

$$\begin{tikzcd} GY \arrow[r, "\alpha_Y"] \arrow[d, "Gf"'] & HY \arrow[d, "Hf"] \\ GX \arrow[r, "\alpha_X"'] & HX \end{tikzcd}$$

We will only consider α-stable maps for the case when $\alpha = \theta : F \longrightarrow Id$ and so we will call them И-stable. These maps will not play a prominent role in this paper but they seem to be important in a larger picture and we will give an indication of this in Section 8. The fact that we need is that every mono is И-stable under certain assumptions that are valid in our context.

LEMMA 2.5. *If F preserves monos and has a* И*-unit θ then the following hold.*

(1) *Every mono is* И*-stable.*
(2) *F preserves pullbacks of monos.*
(3) *F reflects pullbacks of monos.*

Proof. First we show that every mono is И-stable. Let $v : V \rightarrowtail X$ be mono. By hypothesis $Fv : FV \rightarrowtail FX$ is mono. As θ_V is regular epi, we have that $\exists_{\theta_X}(FV) = V$. Finally, as θ_X^* is an isomorphism, its left adjoint $\exists_{\theta_X}$ has to be its inverse so we have that $\theta_X^* V = \theta_X^*(\exists_{\theta_X}(FV)) = FV$.

To show the second item, let $f : Y \longrightarrow X$, $V \rightarrowtail X$ and consider the following rectangle.

$$\begin{tikzcd}
F(f^*V) \arrow[r] \arrow[d, tail] & FV \arrow[r, "\theta_V"] \arrow[d, tail] & V \arrow[d, tail] \\
FY \arrow[r, "Ff"'] & FX \arrow[r, "\theta_X"'] & X
\end{tikzcd}$$

As monos are И-stable, the square on the right is a pullback. So in order to prove that the square in the left also is, we have to prove that the rectangle is a pullback. For this just notice that the rectangle above is the composition of the two pullback squares below, by naturality of θ.

$$\begin{tikzcd}
F(f^*V) \arrow[r, "\theta_{(f^*V)}"] \arrow[d, tail] & f^*V \arrow[r] \arrow[d, tail] & V \arrow[d, tail] \\
FY \arrow[r, "\theta_Y"'] & Y \arrow[r, "f"'] & X
\end{tikzcd}$$

We will not need the third item so we omit the simple proof. □

3. И-relations and the Schanuel Topos

In this section we introduce some notation to deal with И-relations and after a brief presentation of the Schanuel topos we show that it has a И-relation.

LEMMA 3.1. *Let* $\mathbf{D}$ *be a regular category with* $\forall$ *quantifiers. Let* $F : \mathbf{D} \longrightarrow \mathbf{D}$ *be a functor and* $\theta : F \longrightarrow Id$ *be a natural transformation such that for all objects* X*, the map* $\langle F!, \theta_X \rangle : FX \longrightarrow F1 \times X$ *is mono. Then the following are equivalent* (*using the notation presented in Section* 1).

(1) $\theta : F \longrightarrow Id$ *is a* И*-unit.*
(2) *For every* X *and* $U \rightarrowtail \mathbb{A} \times X$ *the following sequent holds.*

$$x \in X \vdash (\forall b \in \mathbb{A})(b \# x \to (b, x) \in U) \leftrightarrow (\exists a \in \mathbb{A})(a \# x \wedge (a, x) \in U).$$

(3) *For all* X *and* $U \rightarrowtail \mathbb{A} \times X$*, the following two sequents hold.*

 (a) $x \in X \vdash (\exists a \in \mathbb{A})(a \# x)$ (Fresh)*,*
 (b) $x \in X \vdash (\exists a \in \mathbb{A})(a \# x \wedge (a, x) \in U) \to (\forall b \in \mathbb{A})(b \# x \to (b, x) \in U)$.

Proof. Items (2) and (3) are easily seen to be equivalent. To prove that (1) and (3) are equivalent consider first item (3a). Its validity is equivalent to the statement that $\pi_X.\zeta_X : \mathbb{A} \# X \longrightarrow X$ is a regular epi. In other words, by Lemma 2.1, that θ_X^* is mono.

On the other hand, item (3b) translates as $\exists_{\pi_X}.\exists_{\zeta_X}.\zeta_X^* \leq \forall_{\pi_X}.\forall_{\zeta_X}.\zeta_X^*$. In other words, $\exists_{\theta_X}.\zeta_X^* \leq \forall_{\theta_X}.\zeta_X^*$. As ζ_X is mono, ζ_X^* is epi and so item (3b) is equivalent to $\exists_{\theta_X} \leq \forall_{\theta_X}$. In turn, by Lemma 2.2 this is equivalent to θ_X^* being epi. So we have that item (3) is equivalent to θ_X^* being an isomorphism. □

Many results in the paper will be stated for a topos with a И-relation and many proofs will use the internal logic of the underlying topos. So, for example, functoriality of $\mathbb{A}\ \#\ (\_)$ will then be used in the form: for every $f : Y \longrightarrow X$ the following sequent holds: $a \in \mathbb{A}, y \in Y \vdash a\ \#\ y \to a\ \#\ fy$. It is also easy to prove that a map $f : Y \longrightarrow X$ is И-stable if and only if the sequent $a \in \mathbb{A}, y \in Y \vdash a\ \#\ fy \to a\ \#\ y$ holds.

We now introduce the Schanuel topos **Sch** following [8]. Fix a countably infinite set $\mathbb{A}$ and let G be the group of bijections from $\mathbb{A}$ to itself with the binary operation given by composition and identity operation given by $id_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}$. For any G-set X, x in X and π in G we denote the action of π on x by $\pi \cdot x$. For any X and x as above we say that a subset $w \subseteq \mathbb{A}$ *supports* x if every $\pi \in G$ that fixes w also fixes x. In other words:

$$(\forall \pi \in G)[(\forall a \in w)(\pi a = a) \to \pi \cdot x = x].$$

We say that x *has finite support* if there exists a finite set that supports x. Although this is not obvious (see Proposition 3.4 in [8]), it is the case that if x has finite support then there is a least set supporting x. This least set is denoted by $\mathbf{supp}\, x$. The category of all those G-sets for which every element has finite support is a well known Boolean topos usually called the *Schanuel topos* (see Corollary III.9.3 in [15] for a different presentation). The terminal object in **Sch** is the unique G-set with one element, say $*$. Clearly, $\mathbf{supp}\, * = \emptyset$. The underlying set of the product $X \times Y$ is the product of the underlying sets and the action is taken pointwise. For any $(x, y) \in X \times Y$ we have that $\mathbf{supp}(x, y) = \mathbf{supp}\, x \cup \mathbf{supp}\, y$. Coproducts are also lifted from **Set** and the action is inherited from the components. There is a monoidal structure $\# : \mathbf{Sch} \times \mathbf{Sch} \longrightarrow \mathbf{Sch}$ that can be described as follows.

$$X\ \#\ Y = \{(x, y) \mid \mathbf{supp}\, x \cap \mathbf{supp}\, y = \emptyset\}.$$

The action on $X\ \#\ Y$ is inherited from the product $X \times Y$ and the unit of the tensor is the terminal object 1. The set $\mathbb{A}$ has an obvious G-set structure such that for every $a \in \mathbb{A}$, $\mathbf{supp}\, a = \{a\}$. So $\mathbb{A}\ \#\ X = \{(a, x) \mid a \notin \mathbf{supp}\, x\}$ and $\mathbb{A}\ \#\ \mathbb{A} = \{(a, b) \mid a \neq b\}$. Notice that $\mathbb{A}$ is not the countable coproduct of the terminal 1. Indeed, $\mathbb{A}$ does not have any point and it is not decomposable as a non-trivial coproduct.

It is now almost trivial to check that $\mathbb{A}\ \#\ (\_) : \mathbf{Sch} \longrightarrow \mathbf{Sch}$ is a И-relation. Indeed, in this case, $\mathbb{A}\ \#\ X$ is defined as a subobject of $\mathbb{A} \times X$ and we can let θ_X be the projection $\mathbb{A}\ \#\ X \rightarrowtail \mathbb{A} \times X \longrightarrow X$. We now check that item (3) of Lemma 3.1 holds. As every x in X has finite support, (3a) clearly holds. Now assume that $(\exists a \in \mathbb{A})(a\ \#\ x \wedge (a, x) \in U)$ and let $b \in \mathbb{A}$ be such that $b\ \#\ x$. Let $\sigma \in G$ be the bijection that permutes a and b and leaves everything else alone. Clearly, σ leaves the support of x alone so $\sigma \cdot x = x$ and then $\sigma \cdot (a, x) = (\sigma \cdot a, \sigma \cdot x) = (b, x) \in U$. So θ_X is a И-unit. It remains to prove that $\mathbb{A} \# (\_) : \mathbf{Sch} \longrightarrow \mathbf{Sch}$ preserves pullbacks. For this consider the following lemma.

LEMMA 3.2. *Let* $\mathbf{D}$ *be a regular category and* $F : \mathbf{D} \longrightarrow \mathbf{D}$ *preserve pullbacks of monos. Then* F *preserves pullbacks if and only if for every* X *and* Y*,* F *preserves the following pullback.*

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "!"] \\ X \arrow[r, "!"'] & 1 \end{tikzcd}$$

Proof. A simple categorical argument not involving И-quantifiers. □

The functor $\mathbb{A} \# (\_)$ is a subfunctor of $\mathbb{A} \times (\_)$ so it must preserve monos. Hence, it preserves pullbacks of monos by Lemma 2.5. Then, to prove that $\mathbb{A} \# (\_) : \mathbf{Sch} \longrightarrow \mathbf{Sch}$ preserves pullbacks, we need only show that it preserves the particular pullbacks stated in Lemma 3.2. The issue then reduces to showing the validity of the following internal statement.

$$a \in \mathbb{A}, x \in X, y \in Y \vdash a \# x \wedge a \# y \to a \# (x, y).$$

But this holds by the construction of products in **Sch**. This finishes the proof that **Sch** has a И-relation.

4. The Right Adjoint to $\mathbb{A} \# (\_)$

In this section we prove Proposition 1.3. For the remaining of the section let $\mathbf{D}$ be a topos with a И-relation $\mathbb{A} \# (\_)$. We denote the classifier of partial maps (with codomain X) by $X_\perp$ (see Section 1.2 in [10]). We have to show that $\mathbb{A} \# (\_) : \mathbf{D} \longrightarrow \mathbf{D}$ has a right adjoint. For any X consider the following definitions.

$$P = \{f \in (X_\perp)^{\mathbb{A}} \mid (\text{И}a)(fa \in X)\} \rightarrowtail (X_\perp)^{\mathbb{A}}.$$

Clearly, we have an "evaluation" map $ev : \mathbb{A} \# P \longrightarrow X$. For any generalized $(a, f) \in P$ we write fa instead of $ev(a, f)$.

$$E = \{(f, g) \in P \times P \mid (\text{И}a)(fa = ga)\} \rightarrowtail P \times P.$$

The relation E is clearly reflexive and symmetric. Now assume that fEg and gEh. By (Fresh) we have $(\text{И}a)(a \# (f, g, h))$ and hence $fa = ga = ha$ which implies $(\text{И}a)(fa = ha)$. So E is transitive and hence an equivalence relation. Let $e : P \twoheadrightarrow [\mathbb{A}]X$ be the (effective) quotient of P by this equivalence relation. As $\mathbb{A} \# (\_)$ is a И-relation, it preserves pullbacks and by Lemma 2.3 it preserves (regular) epis. We then have that $\mathbb{A} \# (\_)$ preserves exact sequences and we have the following diagram.

$$\begin{tikzcd} \mathbb{A} \# E \arrow[r, shift left] \arrow[r, shift right] & \mathbb{A} \# P \arrow[r, two heads, "\mathbb{A} \# e"] \arrow[dr, "ev"'] & \mathbb{A} \# [\mathbb{A}]X \arrow[d, dashed, "\epsilon"] \\ & & X \end{tikzcd}$$

We now show that the map $\epsilon : \mathbb{A} \# [\mathbb{A}]X \longrightarrow X$ is the counit of the adjunction we are looking for. For any $t : \mathbb{A} \# Y \longrightarrow X$ let $t_0 : \mathbb{A} \times Y \longrightarrow X_\perp$ be the classifying map of the partial map $(\zeta_Y, t) : \mathbb{A} \times Y \rightharpoonup X$. Let $t_1 : Y \longrightarrow (X_\perp)^{\mathbb{A}}$ be given by cartesian closure. It is easy to show that t_1 factors through $P \rightarrowtail (X_\perp)^{\mathbb{A}}$ via a map $t_2 : Y \longrightarrow P$. Then consider the map $e.t_2 : Y \longrightarrow [\mathbb{A}]X$. It is easy to show that $\epsilon.(\mathbb{A} \# (e.t_2)) = t : \mathbb{A} \# Y \longrightarrow X$. So we have proved that for any $t : \mathbb{A} \# Y \longrightarrow X$ there exists a $g : Y \longrightarrow [\mathbb{A}]X$ such that $\epsilon.(\mathbb{A} \# g) = t$. Uniqueness of g follows from the lemma below which is what Gabbay and Pitts call "extensionality for atom abstractions" in [8]. In order to state it we will use the notation presented in the introduction: for any $(a, f) \in \mathbb{A} \# [\mathbb{A}]X$ we write fa instead of $\epsilon(a, f)$.

LEMMA 4.1. *The sequent* $f, g \in [\mathbb{A}]X \vdash (Ͷa \in \mathbb{A})(fa = ga) \leftrightarrow f = g$ *holds for every* X. *In other words, the map* $\langle \pi_0.\zeta, \epsilon\rangle : \mathbb{A} \# [\mathbb{A}]X \longrightarrow \mathbb{A} \times X$ *is mono.*

Proof. One direction is trivial, to prove the other let $f, g : Z \longrightarrow [\mathbb{A}]X$ be generalized elements. First, we obtain "representatives" in P.

$$\begin{tikzcd} Z' \arrow[r, "\langle f', g'\rangle"] \arrow[d, two heads, "e'"'] & P \times P \arrow[d, two heads, "e \times e"] \\ Z \arrow[r, "\langle f, g\rangle"'] & {[\mathbb{A}]X \times [\mathbb{A}]X} \end{tikzcd}$$

Validity of $(Ͷa \in \mathbb{A})(fa = ga)$ implies that $\epsilon.(\mathbb{A} \# f) = \epsilon.(\mathbb{A} \# g)$. We then have the following diagram.

$$\begin{tikzcd} \mathbb{A} \# Z' \arrow[r, "{\mathbb{A} \# f', \mathbb{A} \# g'}"] \arrow[d, two heads, "\mathbb{A} \# e'"'] & \mathbb{A} \# P \arrow[r, shift left, "ev"] \arrow[r, shift right] \arrow[d, two heads, "\mathbb{A} \# e"'] & X \\ \mathbb{A} \# Z \arrow[r, "{\mathbb{A} \# f, \mathbb{A} \# g}"'] & \mathbb{A} \# [\mathbb{A}]X \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

It follows that $ev.(\mathbb{A} \# f') = ev.(\mathbb{A} \# g')$ and so, f' is E-related to g'. Hence $f.e' = e.f' = e.g' = g.e'$ and as e' is epi, $f = g$. □

(Below we will denote the map $\langle \pi_0.\zeta, \epsilon\rangle$ simply by $\langle \pi_0, \epsilon\rangle$.) This finishes the proof of Proposition 1.3. Many good properties of the operation $[\mathbb{A}](\_)$ stated in [8] follow from the functor being a right adjoint. Gabbay and Pitts build $[\mathbb{A}]X$ as a quotient of $\mathbb{A} \times X$ by an equivalence relation constructed using И-quantifiers and certain permutation maps $\mathbb{A} \times \mathbb{A} \times X \longrightarrow X$ present in the Schanuel topos. But they also observe that the elements of $[\mathbb{A}]X$ can be represented as partial functions. Let us record this fact in our context.

COROLLARY 4.2. *For any X, the obvious map $j : [\mathbb{A}]X \longrightarrow (X_\perp)^{\mathbb{A}}$ induced by the partial $(\zeta_{[\mathbb{A}]X}, \epsilon) : \mathbb{A} \times [\mathbb{A}]X \rightharpoonup X$ is mono.*

Proof. Clearly j factors through $P \rightarrowtail (X_\perp)^{\mathbb{A}}$ via a map $j' : [\mathbb{A}]X \longrightarrow P$. It is easy to show that $ev.(\mathbb{A} \# j') = \epsilon$. Lemma 4.1 then implies that j' is mono (indeed, a section of $e : P \twoheadrightarrow [\mathbb{A}]X$). □

So we have turned the "extensionality principle" mentioned by Gabbay and Pitts in Proposition 5.5 of [8] into the definition of $[\mathbb{A}](\_)$.

4.1. THE SEQUENT $a \# b \leftrightarrow a \neq b$

We briefly explore the assumption that the diagonal $\Delta : \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ is the complement of the relation $\mathbb{A} \# \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$. In other words, that for all $a, b \in \mathbb{A}$, $a \# b$ if and only if $a \neq b$. (Recall that an object X is said to be *decidable* if the diagonal $\Delta : X \rightarrowtail X \times X$ is complemented.) We show that under this assumption the object $[\mathbb{A}]\mathbb{A}$ is canonically iso to $1 + \mathbb{A}$ and prove that the natural formula to calculate the free variables of a term with a bound variable holds for the elements of $[\mathbb{A}]X$ for any X.

Denote by $\iota : 1 \longrightarrow [\mathbb{A}]\mathbb{A}$ the transposition of the identity $\mathbb{A} \# 1 = \mathbb{A} \longrightarrow \mathbb{A}$. The map ι is obviously mono. On the other hand, denote by $\tau_X : X \longrightarrow [\mathbb{A}]X$ the transposition of $\theta_X : \mathbb{A} \# X \longrightarrow X$. As θ is a natural epi it follows that τ_X is mono for every X. Intuitively, ι is the term $(\lambda a.a)$ while τx is the term $(\lambda b.x)$ where b is some name not appearing free in x.

LEMMA 4.3. *If the diagonal $\Delta : \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ is the complement of the relation $\mathbb{A} \# \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ then the map $[\iota, \tau] : 1 + \mathbb{A} \longrightarrow [\mathbb{A}]\mathbb{A}$ is an isomorphism.*

Proof. Let $f \in [\mathbb{A}]\mathbb{A}$. We show that either $f = \iota$ or $f = \tau b$ for some $b \in \mathbb{A}$. By freshness let $a \# f$. Also, let $b = fa$. By hypothesis $\mathbb{A}$ is decidable and we have that either $a = b$ or $a \# b$. Using Lemma 4.1 it is trivial to show that if $a = b$ then $f = \iota.!$ and if $a \# b$ then $f = \tau a$. So the canonical map $[\iota, \tau] : 1 + \mathbb{A} \longrightarrow [\mathbb{A}]\mathbb{A}$ is epi. Well known properties of coproducts in toposes imply that in order to prove that $[\iota, \tau] : 1 + \mathbb{A} \longrightarrow [\mathbb{A}]\mathbb{A}$ is mono it is enough to show that the subobjects ι and τ of $[\mathbb{A}]\mathbb{A}$ are disjoint. For this assume that $\iota = \tau a$ and let $b \# a$. Then $b = \iota b = (\tau a)b = a$ which is absurd. So ι and τ are disjoint, hence $[\iota, \tau]$ is mono and as it is also epi we can conclude that it is an iso. □

Gabbay and Pitts show that the support of an abstraction $f = (\lambda a.x) \in [\mathbb{A}]X$ is given by $\mathbf{supp}\, f = (\mathbf{supp}\, x)\backslash\{a\}$. We show that this also holds in the present more general context.

LEMMA 4.4. *If* $\Delta : \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ *is the complement of* $\mathbb{A} \# \mathbb{A}$ *then the sequent* $f \in [\mathbb{A}]X \vdash (\text{И}a)(\forall b \in \mathbb{A})(b \# f \leftrightarrow b \# fa \vee b = a)$ *holds.*

Proof. Assume that $a \# f$ and let $b \in \mathbb{A}$. By hypothesis we can concentrate on the case when $a \# b$. Assume first that $b \# f$ we then have that $b \# (a, f)$ and hence that $b \# fa$. To prove the converse notice that by Lemma 4.1 the map $\langle \pi_0, \epsilon \rangle : \mathbb{A} \# [\mathbb{A}]X \longrightarrow \mathbb{A} \times X$ is mono and hence И-stable by Lemma 2.5. Then, if $b \# fa$ and $b \# a$ we have that $b \# (a, fa)$ and so $b \# (a, f)$ by И-stability. It follows that $b \# f$. □

5. Binders and the Right Adjoint to $[\mathbb{A}](\_)$

In this section we first show Lemma 1.5 which says that the existence of pre-binders (Definition 1.4) is equivalent to the existence of a particular right adjoint to $[\mathbb{A}](\_)$. We then study binders (Definition 1.6) and prove that their existence implies the existence of pre-binders. Moreover, under a mild condition on $\mathbb{A} \# \mathbb{A}$ we express binders in terms of certain "name swapping" operations $\mathbb{A} \# (\mathbb{A} \times X) \longrightarrow X$ and finally combine the existence of binders with decidability of $\mathbb{A}$ in order to strengthen the results in Section 4.1.

As before, let $\mathbf{D}$ be a topos with a И-relation $\mathbb{A} \# (\_)$. By Proposition 1.3 we have that $\mathbb{A} \# (\_) \dashv [\mathbb{A}](\_)$. In this context, recall that $\mathbf{D}$ has pre-binders (Definition 1.4.) if it comes equipped with a natural transformation inducing an isomorphism between maps $g : [\mathbb{A}]X \longrightarrow Y$ and maps $f : \mathbb{A} \times X \longrightarrow Y$ such that $a \in \mathbb{A}, x \in X \vdash a \# f(a, x)$. (Recall also that $(\lambda a.x)$ is a suggestive way of writing $\lambda(a, x)$.)

LEMMA 5.1. *Let* $\mathbf{D}$ *be a topos with a* И*-relation* $\mathbb{A} \# (\_) : \mathbf{D} \longrightarrow \mathbf{D}$. *If* $\mathbf{D}$ *has pre-binders given by a natural transformation* $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ *then* λ *is epi and* $a \in \mathbb{A}, x \in X \vdash a \# (\lambda a.x)$ *holds.*

Proof. Trivially λ is epi by the uniqueness condition of pre-binders and since $\lambda = id.\lambda$ it must be the case that the sequent in the statement holds. □

Let $HX = \{f \in X^{\mathbb{A}} \mid (\forall a \in \mathbb{A})(a \# fa)\}$. This definition extends to that of a functor $H : \mathbf{D} \longrightarrow \mathbf{D}$ whose action on maps is given by post composition so that for any $f : X \longrightarrow Y$, $t \in HX$ and $a \in \mathbb{A}$ we have $((Hf)t)a = f(ta)$.

Proof of Lemma 1.5. Assume first that $\mathbf{D}$ has pre-binders. As $a \# (\lambda a.x)$, the transposition of λ factors through $H[\mathbb{A}]X$ so we have map $\eta : X \longrightarrow H[\mathbb{A}]X$ such that $(\eta x)a = (\lambda a.x)$. We now show that η is the unit of the adjunction $[\mathbb{A}](\_) \dashv H$. Let $h : X \longrightarrow HY$. As HY is a subobject of $Y^{\mathbb{A}}$ we can transpose h to obtain an $f : \mathbb{A} \times X \longrightarrow Y$ such that $f(a, x) = (hx)a$. As $hx \in HY$, it follows that $a \# (hx)a = f(a, x)$ and then as λ is a pre-binder there exists a unique $g : [\mathbb{A}]X \longrightarrow Y$ such that $g.\lambda = f$. We then have that $((Hg)(\eta x))a = g((\eta x)a) =$

$g(\lambda a.x) = f(a, x) = (hx)a$. So indeed, $(Hg).\eta = h$. As λ is epi (Lemma 5.1), g is unique.

Assume now that H is right adjoint to $[\mathbb{A}](\_)$ and let $\eta : X \longrightarrow H[\mathbb{A}]X$ be the unit of the adjunction. Much as above, let $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ be the transposition of the map $X \longrightarrow ([\mathbb{A}]X)^{\mathbb{A}}$ induced by the unit. It is easy to see that the universal property of η translates into λ inducing pre-binders. $\square$

Before we go into binders let us state a simple result that will allow us to prove (in Section 7) that there are examples of toposes with И-relations and adjunctions $\mathbb{A} \# (\_) \dashv [\mathbb{A}](\_) \dashv (\_)_{[\mathbb{A}]}$ but without pre-binders.

LEMMA 5.2. *If* **D** *has pre-binders and* $a, b \in \mathbb{A} \vdash a \# b \rightarrow b \# a$ *holds then, for each* X*, there exists a map* $\sigma : \mathbb{A} \# (\mathbb{A} \times X) \longrightarrow X$ *such that the sequent* $a \in \mathbb{A}, x \in X \vdash (\text{И}b)(a \# \sigma(b, a, x))$ *holds.*

Proof. Let σ be the transposition of $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$. By Lemma 5.1 we have for each x and a that $a \# (\lambda a.x)$. For fresh b it follows from the sequent in the statement that $a \# (b, (\lambda a.x))$ and hence, that $a \# (\lambda a.x)b = \sigma(b, a, x)$. $\square$

Now recall that a binder for X (Definition 1.6) is map $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ such that the following two sequents hold.

(1) $a \in \mathbb{A}, x \in X \vdash a \# (\lambda a.x)$,
(2) $a \in \mathbb{A}, x \in X \vdash (\lambda a.x)a = x$.

We now prove some basic properties of binders, we show that existence of binders implies existence of pre-binders and show that **Sch** has binders. The first thing to notice is the following.

LEMMA 5.3. *The following are equivalent.*

(1) *X has a binder.*
(2) *The map* $\langle \pi_0, \epsilon \rangle : \mathbb{A} \# [\mathbb{A}]X \longrightarrow \mathbb{A} \times X$ *is an iso.*
(3) $a \in \mathbb{A}, x \in X \vdash (\exists f \in [\mathbb{A}]X)(a \# f \wedge fa = x)$.

It then follows that for each X *there is at most one binder. That the binder is epi if it exists and that if every object has a binder then they induce a transformation* $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ *natural in* X.

Proof. A binder λ is given by an inverse $\lambda' : \mathbb{A} \times X \longrightarrow \mathbb{A} \# [\mathbb{A}]X$ to $\langle \pi_0, \epsilon \rangle$ followed by the И-unit. So it is clear that (1) and (2) are equivalent. Item (3) says that the map $\langle \pi_0, \epsilon \rangle : \mathbb{A} \# [\mathbb{A}]X \longrightarrow \mathbb{A} \times X$ is a regular epi. It is also mono by Lemma 4.1 an hence it is an iso. It follows that λ is unique and epi. As $\langle \pi_0, \epsilon \rangle$ is natural it follows that the collection of inverses must be natural too and so the result follows. $\square$

As $[\mathbb{A}](\_)$ has a left adjoint, it preserves finite products. For every X and Y, we intend to use the canonical isomorphism $[\mathbb{A}](X \times Y) \cong [\mathbb{A}]X \times [\mathbb{A}]Y$ in connection with the map $\lambda : \mathbb{A} \times X \times Y \longrightarrow [\mathbb{A}](X \times Y)$.

LEMMA 5.4. *For any X and Y the following holds.*

$$a, b \in \mathbb{A}, x, x' \in X, y, y' \in Y \ \vdash \ (\lambda a.x) = (\lambda b.x') \wedge (\lambda a.y) = (\lambda b.y') \\ \rightarrow \ (\lambda a.(x, y)) = (\lambda b.(x', y')).$$

Proof. We give an informal calculation and safely leave the details to the reader: $(\lambda a.(x, y)) = ((\lambda a.x), (\lambda a.y)) = ((\lambda b.x'), (\lambda b.y')) = (\lambda b.(x', y'))$. □

It is relevant to notice that binders interact well with τ and ι.

LEMMA 5.5. *The following squares are pullbacks.*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "!"] \arrow[d, tail, "\Delta"'] & 1 \arrow[d, tail, "\iota"] \\ \mathbb{A} \times \mathbb{A} \arrow[r, two heads, "\lambda"'] & {[\mathbb{A}]\mathbb{A}} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \# X \arrow[r, two heads, "\theta"] \arrow[d, tail, "\zeta"'] & X \arrow[d, tail, "\tau"] \\ \mathbb{A} \times X \arrow[r, two heads, "\lambda"'] & {[\mathbb{A}]X} \end{tikzcd}$$

Hence the following sequents hold.

(1) $a, b \in \mathbb{A} \vdash (\lambda a.a) = (\lambda b.b)$.
(2) $x \in X \vdash (\text{И}a)(\forall b \in \mathbb{A})(b \# x \leftrightarrow b \# (\lambda a.x))$.
(3) $x \in X \vdash (\text{И}a)(\forall b \in \mathbb{A})(b \# x \rightarrow (\lambda a.x)b = x)$.

Proof. It is easy to see that the square on the left commutes since for every $a \in \mathbb{A}$ we have that $a \# (\lambda a.a)$ and $(\lambda a.a)a = a$. To show that it is a pullback, let $\langle a, b\rangle : Z \longrightarrow \mathbb{A} \times \mathbb{A}$ be such that $(\lambda a.b) = \iota.!$. We then have that $b = (\lambda a.b)a = (\iota.!)a = a$. Now consider the square on the right. Let $(a, x) \in \mathbb{A} \# X$. We then have that $a \# (\lambda a.x)$ and that $a \# \tau x$. It is easy to show that $(\text{И}a)((\tau x)a = x)$, together with Definition 1.6 we can calculate $(\lambda a.x)a = x = (\tau x)a$. It follows by Lemma 4.1 that the diagram in the statement commutes. To show that the diagram is a pullback let $\langle a, x'\rangle : Z \longrightarrow \mathbb{A} \times X$ and $x : Z \longrightarrow X$ be such that $\tau x = (\lambda a.x')$. As $a \# (\lambda a.x')$ and τ is И-stable, we have that $a \# x$. We also have that $x' = (\lambda a.x')a = (\tau x)a = x$. Altogether we have that $a \# x = x'$ and it follows that the square on the right is a pullback. Finally, it is easy to show that the sequents hold. □

The following result is essentially Lemma 6.3 in [8]. We give here a slightly different proof in the general context of this paper.

LEMMA 5.6. *If* **D** *has binders then it has pre-binders.*

Proof. By Lemma 5.3 binders form a natural $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$. To prove that they provide **D** with pre-binders let $f : \mathbb{A} \times X \longrightarrow Y$. First assume that there exists a (necessarily unique as λ is epi) $g : [\mathbb{A}]X \longrightarrow Y$ such that $g.\lambda = f$. As $a \# (\lambda a.x)$, we have that $a \# g(\lambda a.x) = f(a, x)$. On the other hand, as λ is epi by Lemma 5.3 we then have that f factors through $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ if and only if $a, b \in \mathbb{A}, x, x' \in X \vdash \lambda(a, x) = \lambda(b, x') \to f(a, x) = f(b, x')$. So assume that $\lambda(a, x) = \lambda(b, x')$. By our results on the naturality of λ (Lemma 5.3) and its behavior with respect to products (Lemma 5.4) we obtain that $(\lambda a.(a, x)) = (\lambda b.(b, x'))$. We can then calculate as follows.

$$\begin{aligned} f(a, x) &= (\lambda a.f(a, x))b \quad a \# f(a, x) \text{ and Lemma 5.5} \\ &= (([\mathbb{A}]f)(\lambda a.(a, x)))b \quad \text{Naturality} \\ &= (([\mathbb{A}]f)(\lambda b.(b, x')))b \quad \text{Lemma 5.4} \\ &= (\lambda b.f(b, x'))b \quad \text{Naturality} \\ &= f(b, x') \quad \text{Binder} \end{aligned}$$ □

The fact that **Sch** has binders follows from the result below.

LEMMA 5.7. *If the sequent $a, b \in \mathbb{A} \vdash a \# b \to b \# a$ holds in* **D** *then X has a binder if and only if there exists a map $\sigma : \mathbb{A} \# (\mathbb{A} \times X) \longrightarrow X$ satisfying the following sequents.*

(1) $a \in \mathbb{A}, x \in X \vdash (\text{И}b \in \mathbb{A})(a \# \sigma(b, a, x))$.
(2) $a \in \mathbb{A}, x \in X \vdash (\text{И}b \in \mathbb{A})(\sigma(a, b, \sigma(b, a, x)) = x)$.
(3) $a \in \mathbb{A}, x \in X \vdash (\text{И}b \in \mathbb{A})(\text{И}c \in \mathbb{A})(\sigma(c, b, \sigma(b, a, x)) = \sigma(c, a, x))$.

Proof. Assume we are given $\sigma : \mathbb{A} \# (\mathbb{A} \times X) \longrightarrow X$, let $\lambda : \mathbb{A} \times X \longrightarrow [\mathbb{A}]X$ be its transposition and let $(a, x) \in \mathbb{A} \times X$ and $b \# (a, x)$. By assumption and preservation of pullbacks we have that $a \# (b, x)$. In order to show that $a \# (\lambda a.x)$ let $c \# (a, b, x)$ and notice that item (3) implies that $(\lambda b.\sigma(b, a, x))c = (\lambda a.x)c$. It follows that $(\lambda b.\sigma(b, a, x)) = (\lambda a.x)$ and item (1) implies that $a \# (\lambda b.\sigma(b, a, x))$ and hence $a \# (\lambda a.x)$. Finally, using item (2) we can calculate $(\lambda a.x)a = (\lambda b.\sigma(b, a, x))a = \sigma(a, b, \sigma(b, a, x)) = x$.

Conversely, we can use Lemma 5.2 to prove sequent (1). To prove that sequents (2) and (3) hold for σ observe that as λ is a binder we have that $(\lambda b.(\lambda a.x)b)b = (\lambda a.x)b$ and we can conclude that $(\lambda b.(\lambda a.x)b) = (\lambda a.x)$. To obtain sequents (2) and (3) just apply this equality to the names a and c respectively. □

So $[\mathbb{A}](\_)$ has a right adjoint in **Sch** and then it preserves colimits. This is a key property in the construction of free algebras needed in [8]. Recall from Section 3 that we have also used the swapping operations in **Sch** in order to show that it has a И-relation (see also Section 6). Yet, the presentation wants to stress that in the general context the operation $[\mathbb{A}](\_)$ can be built without the swapping operations. With this perspective, the swapping operations (and the associated binders) are the simplest way we know to show that $[\mathbb{A}](\_)$ has a right adjoint in **Sch**. This is why Lemma 5.7 was proved at the end of this section. On the other hand, we would like to stress that we believe that the adjoint $(\_)_{[\mathbb{A}]}$ itself is the important notion. When studying other models, the swapping operations (or binders) may not be there but the adjoints should be. We will return to this in Section 7 where, among some examples, we show that binders do not survive slicing while the adjoint $(\_)_{[\mathbb{A}]}$ does.

It may be important to note that in the case of the Schanuel topos, the string $\mathbb{A} \# (\_) \dashv [\mathbb{A}](\_) \dashv (\_)_{[\mathbb{A}]}$ is related to the fact that $\mathbb{A}$ is infinite and decidable together with the role of **Sch** as a classifying topos (see exercises 7, 8 and 9 in Chapter VIII of [15]).

It may also be important to notice that the existence of the maps σ may allow one to build the objects $[\mathbb{A}]X$ in a category weaker than a topos since one could use Gabbay and Pitts' definition. That is, quotienting the object $\mathbb{A} \times X$ by the equivalence relation $(b, x) \sim (c, x') \leftrightarrow (\text{И}a)(\sigma(a, b, x) = \sigma(a, c, x'))$. On the other hand, we are not aware of models whose underlying categories are not toposes.

5.1. BINDERS AND THE DECIDABILITY OF $\mathbb{A}$

In this subsection we combine existence of binders and decidability of $\mathbb{A}$ to strengthen the results in Section 4.1. In particular we give internal description of the embedding $[\mathbb{A}]X \rightarrowtail (X_\perp)^{\mathbb{A}}$ mentioned in Corollary 4.2.

LEMMA 5.8. *If* **D** *has binders then the following are equivalent.*

(1) *The diagonal* $\Delta : \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ *is the complement of* $\mathbb{A} \# \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$.
(2) *The map* $[\iota, \tau] : 1 + \mathbb{A} \longrightarrow [\mathbb{A}]\mathbb{A}$ *is an isomorphism.*

Proof. Lemma 4.3 proves that (1) implies (2). In order to show that (2) implies (1) notice that by assumption we have that the diagram $\iota : 1 \longrightarrow [\mathbb{A}]\mathbb{A} \longleftarrow \mathbb{A} : \tau$ is a coproduct. By Lemma 5.5 we can pullback these injections to obtain the diagram $\Delta : \mathbb{A} \longrightarrow \mathbb{A} \times \mathbb{A} \longleftarrow \mathbb{A} \# \mathbb{A} : \zeta$. As coproducts in toposes are stable under pullback the second diagram is also a coproduct and this is equivalent to 1. □

We now characterize $[\mathbb{A}]X$ internally as a subobject of partial maps.

LEMMA 5.9. *If* **D** *has binders and* $\Delta : \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ *is the complement of* $\mathbb{A} \# \mathbb{A}$ *then* $[\mathbb{A}]X$ *is characterized as a subobject of* $(X_\perp)^{\mathbb{A}}$ *by the following two sequents.*

(1) $f \in (X_\perp)^{\mathbb{A}} \vdash (\forall a)(a \# f \leftrightarrow fa \in X)$.
(2) $f \in (X_\perp)^{\mathbb{A}} \vdash (\text{И}a)(\forall b \in \mathbb{A})(b \# f \leftrightarrow b \# fa \vee b = a)$.

Proof. By Lemma 4.4 every $f \in [\mathbb{A}]X$ satisfies the second sequent. On the other hand, any $f \in [\mathbb{A}]X$ satisfies the left to right implication of the first sequent. To show the converse assume that $fa \in X$ and notice that $a \# (\lambda a.fa) = f$. So $[\mathbb{A}]X$ is a subobject of the object $X' \rightarrowtail (X_\perp)^{\mathbb{A}}$ induced by the two sequents in the statement. It remains to show that the embedding $[\mathbb{A}]X \rightarrowtail X'$ is epi. For this, it is enough to show that for any $f \in X'$ and $a \# f$ we have that $(\lambda a.fa) = f \in (X_\perp)^{\mathbb{A}}$. So we have to show that both elements have the same domain of definition and that their values coincide therein. To show that they have the same domain of definition it is enough to show that the sequent $(\forall b \in \mathbb{A})(b \# (\lambda a.fa) \leftrightarrow b \# f)$ holds. Let $b \in \mathbb{A}$. Clearly, we can concentrate in the case when $a \# b$ and then Lemma 4.4 can be used to show that the "support" of $(\lambda a.fa)$ is that of f. Finally, in order to show f and $(\lambda a.fa)$ coincide in their domain, it is enough to show that $(\text{И}b)(fb = (\lambda a.fa)b)$. But we have that $(\lambda a.fa)a = fa$. □

6. Nominal Logic

In this section readers are assumed to be familiar with Pitts' [18]. We introduce the notion of a *nominal relation* in a regular category in order to relate Pitts' presentation with the results in the previous sections. Except for (P1) and (P2) the axioms in Definition 6.1 are named so that they can be easily related to Pitts' axioms stated in Figure 1 of [18].

DEFINITION 6.1. Let **D** be a regular category. A *nominal relation* is given by the following data.

(1) An object $\mathbb{A}$.
(2) For each object X in **D**, a relation $\mathbb{A} \# X \rightarrowtail \mathbb{A} \times X$ such that the following hold.
   - (P1) $a \in \mathbb{A}, x \in X, y \in Y \vdash a \# (x, y) \leftrightarrow a \# x \wedge a \# y$.
   - (F2) The relation $\mathbb{A} \# \mathbb{A}$ is the complement of the diagonal.
   - (F4) The sequent (Fresh) holds.
(3) For each X, a map $\sigma : \mathbb{A} \times \mathbb{A} \times X \longrightarrow X$ such that the following hold.
   - (P2) For every X and Y the sequent below holds
$$(x, y) \in X \times Y, a, a' \in \mathbb{A} \vdash \sigma(a', a, (x, y)) \\ = (\sigma(a', a, x), \sigma(a', a, y)).$$
   - (E1-4) $a, a' \in \mathbb{A}, x \in X \vdash \sigma(a', a, fx) = f(\sigma(a', a, x))$ for every $f : X \longrightarrow Y$.
   - (F1) $a, a' \in \mathbb{A}, x \in X \vdash a \# x \wedge a' \# x \rightarrow \sigma(a', a, x) = x$.
   - (S1) $a \in \mathbb{A}, x \in X \vdash \sigma(a, a, x) = x$.
   - (S2) $a, a' \in \mathbb{A}, x \in X \vdash \sigma(a, a', \sigma(a', a, x)) = x$.
   - (S3) $a, a' \in \mathbb{A}, x \in X \vdash \sigma(a', a, a) = a'$.

(The implication in (F1) presents no problem as it can be formulated in **D** by a commutative diagram.) Pitts introduces Nominal Logic as a theory in many-sorted first-order logic. In his presentation there can be more than one sort of atoms but here we restrict to just one and this is why Pitts' axiom (F3) is not present. So there is a distinguished sort of atoms $\mathbb{A}$ and for each sort X a distinguished relation symbol $\mathbb{A} \# X$ of arity $\mathbb{A}, X$ and a distinguished function symbol with arity

$\mathbb{A}, \mathbb{A}, X \longrightarrow X$ whose effect on terms a, a' of sort $\mathbb{A}$ and x of sort X is written as $(a'a) \cdot x$. The parallel with the definition of a nominal relation is clear, we write $\sigma(a', a, x)$ instead of $(aa') \cdot x$. In notes 2 and 3 of Figure 1 in [18] it is stated that for a finite list of variables $\vec{x}$, $a \# \vec{x}$ indicates the finite list of arguments given by $a \# x_i$ for each x_i in $\vec{x}$ and similarly for $(a'a) \cdot \vec{x}$. This is captured by conditions (P1) and (P2). Axioms (E1) to (E4) in [18] are essentially stating that σ is a natural transformation. But in that context this condition has to be split into (E1) which says that σ is natural with respect to σ itself (which is a distinguished function symbol), (E2) which is the case of the distinguished relations $\mathbb{A} \# (\_)$ and (E3) and (E4) dealing with the rest of function and relation symbols. In the context of a regular category this distinctions are not necessary and that is why we only have condition (E1-4). So it is fair to say that nominal relations are essentially the same thing as Nominal Logic in a regular category. With this in mind, Proposition 4.3 in [18] says that the family of relations $\mathbb{A} \# (\_)$ is actually a functor. If we denote by $\theta_X : \mathbb{A} \# X \longrightarrow X$ the projection onto X of the distinguished relation $\mathbb{A} \# X \rightarrowtail \mathbb{A} \times X$ then it is easy to see that the following holds.

LEMMA 6.2. *A nominal relation is given by a functor* $\mathbb{A} \# (\_) : \mathbf{D} \longrightarrow \mathbf{D}$ *and natural transformations* $\theta : \mathbb{A} \# (\_) \longrightarrow Id$ *and* $\sigma : \mathbb{A} \times \mathbb{A} \times Id \longrightarrow Id$ (*where* $\mathbb{A} = \mathbb{A} \# 1$) *such that the following hold.*

(1) $\mathbb{A} \# \mathbb{A} \rightarrowtail \mathbb{A} \times \mathbb{A}$ *is the complement of the diagonal.*
(2) *The map* $\langle \mathbb{A} \# !, \theta \rangle : \mathbb{A} \# X \longrightarrow \mathbb{A} \times X$ *is mono and* (Fresh) *holds.*
(3) *F preserves the pullbacks displayed in Lemma* 3.2.
(4) *σ satisfies* (F1), (S1), (S2) *and* (S3).

Now suppose we have a topos $\mathbf{D}$ with a nominal relation $\mathbb{A} \# (\_)$. Proposition 5.1 in [18] says that item (3b) of Lemma 3.1 holds. So $\mathbb{A} \# (\_)$ is a И-relation. The reader is invited to show that the axioms for a nominal relation imply that the obvious restriction $\mathbb{A} \# (\mathbb{A} \times X) \longrightarrow X$ of σ satisfies the items of Lemma 5.7. In this way it is fair to conclude that Nominal Logic in a topos $\mathbf{D}$, provides $\mathbf{D}$ with a И-relation $\mathbb{A} \# (\_)$, binders and such that $\mathbb{A} \# \mathbb{A}$ is the complement of the diagonal.

7. Other И-units

In this section we discuss new examples of toposes with И-relations. First we show that И-relations are inherited by slices and by the arrow construction (which provides examples of И-relations with non-boolean underlying toposes). We then briefly discuss (without proofs) how further examples arise as Kleisli categories. For the rest of the section let $F : \mathbf{C} \longrightarrow \mathbf{C}$ be a functor on a regular category and let $\theta : F \longrightarrow Id$ be a И-unit. Let us consider first the case of slices.

For any X, let us denote by $\Delta : \mathbf{C} \longrightarrow \mathbf{C}/X$ the functor that for each Y, ΔY is the projection $X \times Y \longrightarrow X$. Define $F_X : \mathbf{C}/X \longrightarrow \mathbf{C}/X$ to be the functor that to each object $f : Y \longrightarrow X$ in $\mathbf{C}/X$ assigns $f.\theta_Y = \theta_X.(Ff)$. Fix the object X and notice that θ_Y induces a map $(F_X f) \longrightarrow f$ in $\mathbf{C}/X$ and in this way, it induces a natural transformation $\Theta : F_X \longrightarrow Id_{\mathbf{C}/X}$. Recall that $\Delta : \mathbf{C} \longrightarrow \mathbf{C}/X$ has a left adjoint $\Gamma_X : \mathbf{C}/X \longrightarrow \mathbf{C}$ given by $\Gamma f = Y$ for $f : Y \longrightarrow X$ and that there is a natural isomorphism $Sub_{\mathbf{C}/X} f = Sub_{\mathbf{C}}(\Gamma f)$. Moreover any $h : X \longrightarrow Z$ induces a functor $(h.(\_)) : \mathbf{C}/X \longrightarrow \mathbf{C}/Z$ such that $\Gamma_Z.(h.(\_)) = \Gamma_X$ and the functor $F : \mathbf{C} \longrightarrow \mathbf{C}$ induces a functor (denoted with the same letter) $F : \mathbf{C}/X \longrightarrow \mathbf{C}/FX$ such that $\Gamma_{FX}.F = F.\Gamma_X : \mathbf{C}/X \longrightarrow \mathbf{C}$.

LEMMA 7.1. *The natural transformation Θ is a* И*-unit for F_X.*

Proof. Just observe the following sequence of natural isomorphisms.

$$\begin{aligned}
Sub_{\mathbf{C}/X}(f : Y \longrightarrow X) &\cong Sub_{\mathbf{C}}(\Gamma_X f) \\
&\cong Sub_{\mathbf{C}}(F(\Gamma_X f))\ \theta \text{ is a И-unit} \\
&\cong Sub_{\mathbf{C}}(\Gamma_{FX}(Ff)) \\
&\cong Sub_{\mathbf{C}}(\Gamma_X(\theta_X.(Ff))) \\
&\cong Sub_{\mathbf{C}}(\Gamma_X(F_X f)) \\
&\cong Sub_{\mathbf{C}/X}(F_X f).
\end{aligned}$$

So $\Theta^* : Sub_{\mathbf{C}/X}(f : Y \longrightarrow X) \longrightarrow Sub_{\mathbf{C}/X}(F_X f)$ is an iso. □

When F preserves monos it induces (for every X) a monotone map (that we denote with the same letter) $F : Sub(X) \longrightarrow Sub(FX)$. The fact that every mono is И-stable implies that we can write FV instead of $\theta_X^* V$, for every $V \in Sub(X)$. And then we can calculate using the following rules $F \dashv \text{И} \dashv F$. Notice also that for every $U \in Sub(FX)$ we have that $F(\text{И}U) = \theta_X^*(\text{И}U) = U$ and for every $V \in Sub(X)$ that $V = \text{И}(FV)$. Moreover, preservation of monos (and regular epis) implies that for any map $f : Y \longrightarrow X$ and $W \in Sub(Y)$ we can use the equation $\exists_{Ff}(FW) = F(\exists_f W)$ in calculations. Now consider the arrow category $\mathbf{C}^{\rightarrow}$. Any functor $F : \mathbf{C} \longrightarrow \mathbf{C}$ lifts to a functor $F^{\rightarrow} : \mathbf{C}^{\rightarrow} \longrightarrow \mathbf{C}^{\rightarrow}$ and any natural transformation $\theta : F \longrightarrow Id$ lifts to one $\Theta : F^{\rightarrow} \longrightarrow Id_{\mathbf{C}^{\rightarrow}}$.

LEMMA 7.2. *If $F : \mathbf{C} \longrightarrow \mathbf{C}$ preserves monos and θ is a* И*-unit for F then the map $\Theta : F^{\rightarrow} \longrightarrow Id_{\mathbf{C}^{\rightarrow}}$ is a* И*-unit.*

Proof. First notice that for every $v : V \rightarrowtail X$ and $f : Y \longrightarrow X$, Ff factors through Fv if and only if f factors through v. Indeed:

$$\exists_f Y \leq V \leftrightarrow \exists_f Y \leq \text{И}(FV) \leftrightarrow F(\exists_f Y) \leq FV \leftrightarrow \exists_{Ff} FY \leq FV.$$

Then, to prove the lemma we just observe the following sequence of natural isomorphisms.

$$\begin{aligned}
Sub_{\mathbf{C}^{\rightarrow}}(f : Y \longrightarrow X) &\cong \{(K, L) \in Sub_{\mathbf{C}}(Y) \times Sub_{\mathbf{C}}(X) \mid fK \leq L\} \\
&\cong \{(U, V) \in Sub_{\mathbf{C}}(FY) \times Sub_{\mathbf{C}}(FX) \mid (Ff)U \leq V\} \\
&\cong Sub_{\mathbf{C}^{\rightarrow}}(F^{\rightarrow} f).
\end{aligned}$$

So $\Theta^* : Sub_{\mathbf{C}/X}(f : Y \longrightarrow X) \longrightarrow Sub_{\mathbf{C}/X}(F^{\rightarrow} f)$ is an isomorphism. □

We now extend this result to И-relations.

LEMMA 7.3. *If $\theta : F \longrightarrow Id$ is a И-relation in $\mathbf{C}$ then so are the induced И-units in $\mathbf{C}^{\rightarrow}$ and $\mathbf{C}/X$, for any X.*

Proof. Consider first the case of slices. For any $f : Y \longrightarrow X$ as an object in $\mathbf{C}/X$ we have that $F_X 1 \times f$ is given by the pullback of f along $F_X 1 = \theta_X$. In order to show that Θ is a И-relation we first show that for every map $f : Y \longrightarrow X$, the map $\langle Ff, \theta_Y\rangle : FY \longrightarrow FX \times Y$ is mono. But this is easy because $(F! \times id).\langle Ff, \theta_Y\rangle = \langle F!, \theta_Y\rangle$ is mono by hypothesis. Pullbacks in slices are calculated as in the original category. So if $F : \mathbf{C} \longrightarrow \mathbf{C}$ preserves pullbacks then the induced functor on slices also preserve pullbacks. The case of the arrow category is also easy and left for the reader. □

Let us denote the induced "objects of names" by $\mathbb{B}$. And hence together with Lemma 7.3 we obtain right adjoints by Proposition 1.3. Let us call them $[\mathbb{B}](\_)$. In the case of $\mathbf{C}^{\rightarrow}$ it is not difficult to show that $[\mathbb{B}](\_)$ is just $[\mathbb{A}](\_)$ acting on maps in the obvious way. For the case of slices see Proposition 7.5 below. It is also easy to see that if $\mathbb{A}$ is decidable and $\mathbb{A} \# \mathbb{A}$ is $\neg\Delta$ then so is the case in slices and in the arrow category.

LEMMA 7.4. *If $\mathbf{D}$ has binders then so does $\mathbf{D}^{\rightarrow}$.*

Proof. This is an easy consequence of naturality of σ. □

On the other hand, slices do not inherit binders. Let us go through an example. Let $\mathbf{D}$ be the topos $\mathbf{Sch}$ and let $\mathbf{E}$ be the topos $\mathbf{Sch}/\mathbb{A}$. Let $\mathbb{C}$ be the object $\pi_0 : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ so that we think of $\mathbb{C}$ as the object of pairs (a, c) indexed by a. The object of names $\mathbb{B}$ is given by $\theta : \mathbb{A} \# \mathbb{A} \longrightarrow \mathbb{A}$ so we picture it as the object of pairs (d, a) such that $d \# a$ indexed by a. We then have that the product $\mathbb{B} \times \mathbb{C}$ is the object of 3-uples (d, a, e) such that $d \# a$ (indexed by a) and that $\mathbb{B} \# \mathbb{C}$ is the subobject of $\mathbb{B} \times \mathbb{C}$ given by those 3-uples (d, a, e) such that $d \# (a, e)$. Also, the object $\mathbb{B} \# (\mathbb{B} \times \mathbb{C})$ in $\mathbf{E}$ is given by the object of 4-uples (b, d, a, c) such that $(d, a, c) \in (\mathbb{B} \times \mathbb{C})$ and $b \# (d, a, c)$ again indexed by a. Now assume that there is a map $\sigma : \mathbb{B} \# (\mathbb{B} \times \mathbb{C}) \longrightarrow \mathbb{C}$ such that for each $(\beta, \alpha, \gamma) \in \mathbb{B} \# (\mathbb{B} \times \mathbb{C})$ we have that $\alpha \# \sigma(\beta, \alpha, \gamma)$. This means that there exists a map σ that maps a 4-uple (b, d, a, c) (which has index a) as above to an element $\sigma(b, d, a, c)$ in $\mathbb{A} \times \mathbb{A}$ with index a and such that $a \# \sigma(b, d, a, c)$. But this is absurd because $\sigma(b, d, a, c)$ has index a and so must be of the form (a, e) for some e in $\mathbb{A}$. It follows by Lemma 5.2 that $\mathbf{E}$ can not have pre-binders.

One can argue that one of the main interest of (pre-)binders resides on the fact that they allow to build a sort-of-amazing right adjoint. The existence of these adjoints *is* inherited by slices. This observation is a very simple adaptation of what is known as *Johnstone's description of the amazing right adjoint* [12].

PROPOSITION 7.5 (Johnstone). *Let* $\mathbf{C}$ *be a category with finite limits and let* $F \dashv G : \mathbf{C} \longrightarrow \mathbf{C}$ *be an adjunction on it. Let* X *be an object in* $\mathbf{C}$ *and let* $\gamma : FX \longrightarrow X$ *be a map. Then the following hold.*

(1) *The functor* $\gamma.(F(\_)) : \mathbf{C}/X \longrightarrow \mathbf{C}/X$ *has a right adjoint* G_X.
(2) *If, moreover,* G *has a further right adjoint* H *and* $\mathbf{C}$ *is locally Cartesian closed then* G_X *has a right adjoint.*

Proof. Let $f : Y \longrightarrow X$, $g : Z \longrightarrow X$ and let $\rho : X \longrightarrow GX$ be the transposition of γ. Then we have $\mathbf{C}/X(\gamma.Ff, g) \cong \mathbf{C}/GX(\rho.f, Gg) \cong \mathbf{C}/X(f, \rho^*(Gg))$. That is G_X is $\rho^*(G(\_))$. For the second item, let $h : W \longrightarrow X$ and let $u : X \longrightarrow HGX$ be the unit of the adjunction $G \dashv H$. Then we can calculate

$$\begin{aligned}\mathbf{C}/X(\rho^*(Gg), h) &\cong \mathbf{C}/GX(Gg, \Pi_\rho h) \cong \mathbf{C}/X(u.g, H\Pi_\rho h)\\ &\cong \mathbf{C}/X(g, u^*(H\Pi_\rho h)).\end{aligned}$$

So G_X has a right adjoint. □

Recall the notation introduced before Lemma 7.3.

COROLLARY 7.6. *If* $\mathbf{D}$ *has a И-relation* $\mathbb{A} \# (\_)$ *such that* $[\mathbb{A}](\_)$ *has a right adjoint then for every* X *in* $\mathbf{D}$, *the induced functor* $[\mathbb{B}](\_)$ *on the slice* $\mathbf{D}/X$ *also has a right adjoint.*

Proof. Let γ be $\theta_X : \mathbb{A} \# X \longrightarrow X$ in Proposition 7.5. □

We now discuss a family of examples that arise as Kleisli categories. Let $\mathbf{B}$ denote now the (essentially small) groupoid of finite sets and bijections and consider the topos $\mathbf{Set}^{\mathbf{B}}$ which we denote by **Joy**. The inclusion of $\mathbf{B}$ into the category $\mathbf{I}$ of finite sets and monomorphisms induces a monad M on **Joy**. In [6] Fiore observed that the Kleisli category for M is equivalent to the Schanuel topos. Let us make here more explicit the relation between **Joy** and **Sch** in order to motivate how other toposes with И-relations arise in this way.

The subcategory of **Joy** induced by the functors that take values in finite sets is presented in [11] as a general framework for enumerative combinatorics of labeled structures (see also [2]). The idea is that a functor $F \in$ **Joy** (that we may call a *species*) takes a finite set U of labels and produces a set FU of structures labeled with elements of U. If one is interested in counting structures then it is useful to think of an object F of **Joy** as the formal power series below with coefficients in cardinalities.

$$F = F0 + (F1)x + (F2)\frac{x^2}{2} + \cdots + (Fn)\frac{x^n}{n!} + \cdots.$$

Day's construction [4] applied to the operation of disjoint union in $\mathbf{B}$ gives a tensor $\# : \mathbf{Joy} \times \mathbf{Joy} \longrightarrow \mathbf{Joy}$ (actually, a closed structure) which provides a combinatorial interpretation of the product of series. Via Yoneda, the set with a unique element in **B** gives an object that we denote by $\mathbb{A}$ and whose representing series is the single variable x. When series are ordered lexicographically, the series x becomes and infinitesimal (in the sense that it is above 0 yet below every constant) so it may not be surprising that the right adjoint to $\mathbb{A} \# (\_) : \mathbf{Joy} \longrightarrow \mathbf{Joy}$ (that we denote by $\langle\mathbb{A}\rangle(\_) : \mathbf{Joy} \longrightarrow \mathbf{Joy}$) behaves as a derivative operator. Indeed, $\langle\mathbb{A}\rangle(\_)$ can be defined by $(\langle\mathbb{A}\rangle F)U = F(U+1)$ so the representing series of $\langle\mathbb{A}\rangle F$ is the following

$$F1 + (F2)x + \cdots + F(n+1)\frac{x^n}{n!} + \cdots.$$

The object $\mathbb{A}$ in **Joy**, seen as an object in the Kleisli category **Sch**, corresponds to the object of names that we discussed in the previous sections. The tensor $\# : \mathbf{Joy} \times \mathbf{Joy} \longrightarrow \mathbf{Joy}$ lifts to the Kleisli category **Sch** and the functor $\mathbb{A} \# (\_) : \mathbf{Sch} \longrightarrow \mathbf{Sch}$ gives the underlying functor of the И-relation we discussed before.

It is possible to generalize this picture in order to provide more examples of toposes arising as Kleisli categories of monads on "toposes for combinatorics" and examples of "infinitesimals" therein inducing И-relations in the Kleisli category. Indeed let **E** be any essentially small boolean category **E** with finite coproducts, finite limits, effective unions and such that there is no infinite chain of proper subobjects. Let **C** be the subgroupoid induced by the isomorphisms of **E**. It is possible to show that the topos $\mathbf{Set}^{\mathbf{C}}$ comes equipped with a monad M whose Kleisli category is a topos and such that every object C of **C** induces a И-relation in the Kleisli category of M. For example, the topos of *Partitionals* [16] can be quickly described as analogous to **Joy** but with the representing series looking as follows.

$$\sum_{n\geq 0}\ \sum_{(\lambda_1+\cdots+\lambda_k+\cdots=n)} a_{\lambda_1,\lambda_2,\ldots}\frac{x_1^{\lambda_1}x_2^{\lambda_2}\ldots}{1!^{\lambda_1}\lambda_1!2!^{\lambda_2}\lambda_2!\ldots}.$$

Each x_i induces a И-relation in the Kleisli category for certain monad on this topos. The details of these results will be presented elsewhere.

8. Conclusions and Further Developments

The idea of И-quantifiers was presented by Gabbay and Pitts in [8] as a phenomenon occurring in the Fraenkel–Mostowski permutation model of set theory. It was convincingly demonstrated that the idea is useful, for example, by setting up a theory of inductively defined sets that "can correctly model α-equivalence classes of variable binding syntax" and also by designing a programming language incorporating these ideas [19]. Having recognized И-quantifiers as an interesting and useful idea it seemed important to understand it in a level of generality and abstraction that allows the idea to be interpreted in other contexts and its properties to be proved independently of any particular instance. For the usual ∀ and ∃ quantifiers such a treatment is provided by understanding them as adjoints to substitution [13, 17]. We presented such a treatment for И-quantifiers via the notions of И-units in Section 2. Our account provides abstract versions of the results and constructions in [8]. This perspective has also facilitated the construction of new examples of И-quantifiers.

As explained in [8], there are a number of other approaches to deal with syntax involving variable binders. For example, consider the categories and triposes discussed in [7] and [9]. It may be interesting to investigate if those proposals can profit from the analysis made here. In this respect it seems relevant to note that in all these proposals the operation creating a "set of abstractions" arises both as a left and right adjoint. The relation between *Higher Order Abstract Syntax* (HOAS) and the categorical approaches has been studied in [9] and it deserves further study. Both the categorical and HOAS approaches should profit from such research. On the other hand, the theory of И-quantifiers may be pursued for its own sake. For example, let us briefly consider here a topos $\mathbf{D}$ with a И-relation with the extra assumption that for every X the subobject $\mathbb{A} \# X \rightarrowtail \mathbb{A} \times X$ is complemented. Intuitively this is saying that we can decide whether a name appears free in a term or not. Under this assumption it is possible to reconstruct abstractly part of the picture discussed in the end of Section 7. Indeed, it is possible to show that the subcategory of $\mathbf{D}$ induced by the И-stable maps is a topos equipped with a tensor structure # and a monad M whose Kleisli category is the topos $\mathbf{D}$. The details of these results will be discussed elsewhere. As another example of further developments in the "pure" theory of И-quantifiers consider the extra right adjoint $(\_)_{[\mathbb{A}]}$. We used it to show that $[\mathbb{A}](\_)$ preserves colimits (which is a key point in the construction of free algebras) but the adjoint itself is slightly mysterious and we believe that it deserves further exploration. (The objects A such that $(\_)^A$ has a right adjoint have been shown to be quite useful and have received considerable attention, see, e.g., [12].) Already present in the Schanuel topos there are a number of variations on operations that build abstractions which should be studied more closely. Some of them were hinted at in the conference version of [8].

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