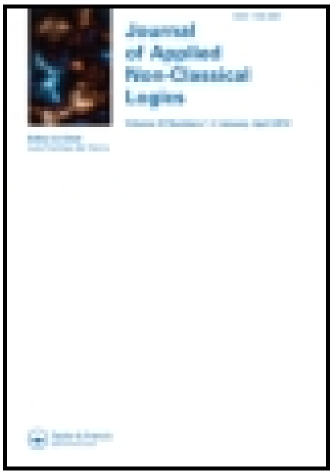

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Compatible operations on commutative residuated lattices

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Compatible operations on commutative residuated lattices

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ABSTRACT. *Let L be a commutative residuated lattice and let $f : L^k \to L$ a function. We give a necessary and sufficient condition for f to be compatible with respect to every congruence on L. We use this characterization of compatible functions in order to prove that the variety of commutative residuated lattices is locally affine complete. Then, we find conditions on a not necessarily polynomial function $P(x, y)$ in L that imply that the function $x \mapsto min\{y \in L \mid P(x, y) \leq y\}$ is compatible when defined. In particular, $P_n(x, y) = y^n \to x$, for natural number n, defines a family, S_n, of compatible functions on some commutative residuated lattices. We show through examples that S_1 and S_2, defined respectively from P_1 and P_2, are independent as operations over this variety;* i.e. *neither S_1 is definable as a polynomial in the language of L enriched with S_2 nor S_2 in that enriched with S_1.*

KEYWORDS: *commutative residuated lattices, compatible operations, affine completeness.*

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1. Introduction

A commutative residuated lattice (CRL) is a distributive lattice $(A, \vee, \wedge)$ equipped with a commutative monoid structure $(A, \cdot, e)$ such that for every x in A, the operation $x \cdot (\_) : A \to A$ is monotone (with respect to the partial order induced by the lattice structure) and has a right adjoint, usually denoted by $x \to (\_)$. This is the situation, for example, when we consider the Heyting implication on the reduct $(A, \wedge, 1)$ of a Heyting algebra. The CRLs form a variety (see section 1, (Hart *et al.*, 2002)).

The intuitionistic logic was introduced as a sequent system by Gentzen in 1934 (see (Gentzen, 1934)). The substructural logics are obtained by dropping from that system some or all of the structural rules: exchange, weakening and contraction (see (Ono, 2003)). There is also some axiomatic versions of that systems, thus classical, intuitionistic, multi-valued, basic, relevant and fragments of linear logic can be considered as substructural logics. It is well known that, by an adequate definition of consequence operator, this logics are algebrizable in the sense of Blok-Pigozzi (see (Ono, 2003), (Blok *et al.*, 1989)). Then, for each substructural logic there is an equivalent algebraic semantics, which happens to be some subvariety of that of residuated lattices.

The problem of adding connectives to extend a logic in a "natural" way has been studied for a long time. For intuitionistic calculus the paper (Caicedo *et al.*, 2001) of Caicedo and Cignoli emphasizes the algebraic aspect of the problem through the notion of *compatible* function, that translates to the notion of compatible connective in intuitionistic logic. In an algebra A an n-ary operation f will be called *compatible* if every congruence of A is a congruence of A enriched with f as a new operation. We can say that a connective k is compatible if and only if $A \leftrightarrow B \vdash kA \leftrightarrow kB$, for all formulas A, B. Results in (Caicedo *et al.*, 2001) are extended to algebrizable logics by Caicedo in (Caicedo, 2004). But it is not clear what can be considered a natural or good extension of a substructural logic by means of new connectives. In section 3 of our paper we give an algebraic approach to bring light to that problem, following basically the characterization of (Caicedo *et al.*, 2001) of compatible functions by means of the relationship between congruences and some particular (convex) subalgebras.

In many varieties it is simpler to study subalgebras of some kind instead of congruences. For instance, to characterize a finitely generated subalgebra is in general easier than to describe principal congruences. This kind of treatment is possible in those varieties in which there is an order-preserving bijection between congruences and convex normal subalgebras. The commutative residuated lattices form such a variety.

A variety has *equationally definable principal congruences*, briefly EDPC, if there is a (finite family of) quaternary $\mathcal{V}$-terms $\{u_i\}_i$, $\{v_i\}_i$ such that for any principal congruence $\Theta_{(a,b)}$, $x\,\Theta_{(a,b)}\,y$ if and only if $u_i(a,b,x,y) = v_i(a,b,x,y)$ for every $i = 1, ..., n$ ((Blok *et al.*, 1989), (Blok *et al.*, n.d.)). This property is also of logical interest because a logic has some kind of deduction theorem if and only if the corresponding variety (obtained by the process of algebraization) has EDPC.

The compatible functions are closely related with finitely generated subalgebras or, alternatively, to principal congruences. Indeed, if the variety $\mathcal{V}$ has EDPC with terms u, v then we can characterize compatible functions in the following way (see (Caicedo *et al.*, 2001)): f is compatible if and only if $u(x, y, f(x), f(y)) = v(x, y, f(x), f(y))$ (for f unary). We show in this paper that in some cases the characterization is still possible even if we have on $\mathcal{V}$ a weaker condition of the type "there exist $\mathcal{V}$-terms

$\{u_{in}\}_{i\in I,n\in\mathbb{N}}$, $\{v_{in}\}_{i\in I,n\in\mathbb{N}}$, with I a finite set such that, for any principal congruence $\Theta_{(a,b)}$, $x\,\Theta_{(a,b)}\,y$ if and only if there is a natural number n such that $u_{in}(a,b,x,y) = v_{in}(a,b,x,y)$, for every $i \in I$".

A function obtained by composition of basic operations of the algebra and parameters (polynomial function) is compatible in every variety. Are there compatible functions different from polynomials? In the variety $\mathcal{B}$ of Boolean algebras the answer is no (Kaarli *et al.*, 2001). We say that $\mathcal{B}$ is an *affine complete variety*. On the other hand, the successor $S(x)$ is a compatible Heyting function which is different from every polynomial. However, the variety of Heyting algebras is *locally* affine complete in the sense that any restriction of a compatible function to a finite subalgebra is a polynomial. We prove in this paper that this is also the case for the variety of CRLs. We define a notion of successor on CRLs, providing examples in MV-algebras and ℓ-groups, and we construct a big family of "successors of order n" and more general compatible functions in a way that generalizes the successor construction of (Caicedo *et al.*, 2001). Moreover we show that neither the variety of CRLs with successor nor the variety of CRL with a successor of order two are affine complete.

2. Preliminary results on CRLs

We start by recalling the definition of a commutative residuated lattice (Hart *et al.*, 2002).

DEFINITION 1. — *A* commutative residuated lattice *(CRL for short) is an algebra* $(L,\wedge,\vee,\cdot,\rightarrow,e)$*, where* $(L,\wedge,\vee)$ *is a lattice,* $(L,\cdot,e)$ *is a commutative monoid, and* $\rightarrow$ *is a binary operation (the* residual *for the monoidal operation) such that for every* x *in L,* $x\cdot(\_), x\rightarrow(\_) : L\rightarrow L$ *are monotone and* $x\cdot(\_)$ *is left adjoint to* $x\rightarrow(\_)$*.*

More explicitely, the adjointess condition says that for any y and z in L,

$$x\cdot y\leq z \quad\Leftrightarrow y\leq x\rightarrow z$$

which are also equivalent to $x\leq y\rightarrow z$ by commutativity of $(L,\cdot,e)$.

Put differently, a CRL is a distributive lattice (seen as a category) equipped with a symmetric monoidal closed structure (Mac Lane, 1971). It is not difficult to check that the class of commutative residuated lattices is a variety (see (Hart *et al.*, 2002)).

Examples of CRLs are Heyting algebras and abelian ℓ-groups where the monoid operation is $\wedge$ and $+$ respectively. Other examples are BL-algebras, MV-algebras and product algebras, among others.

We shall now recall from (Hart *et al.*, 2002) and (Jipsen *et al.*, 2002) many results concerning the structure of the congruence lattice of any CRL, which we shall need later.

We say that a subset C of a poset $(P,\leq)$ is *convex* in P if, whenever $a,b\in C$, then $\{x\in P \mid a\leq x\}\cap\{x\in P\mid x\leq b\}\subseteq C$. We refer to a subset H of a CRL L as being a *subalgebra* of L provided that H is closed with respect to all the operations defined on L. We write $\mathrm{Sub}_C L$ for the set of all convex subalgebras of L, partially ordered by set-inclusion. $\mathrm{Sub}_C L$ is a complete lattice with set-intersection as meet. We write $CS_L[x]$ for the convex subalgebra of L generated by $\{x\}$.

We let $\mathrm{Con}L$ denote the lattice of congruence relations for a CRL L, which is known to be a distributive lattice (Hart *et al.*, 2002).

For all $\theta \in \mathrm{Con}L$, let $\theta(e)$ be the equivalence class of e. Lemma 2.1 in (Hart *et al.*, 2002) states that $\theta(e)$ is a convex subalgebra of L. On the other hand, for any convex subalgebra H of L, set

$$\theta_H := \{(x, y) \in L \times L \mid \; x \cdot h \leq y \text{ and } y \cdot h \leq x, \text{ for some } h \in H\}$$
$$:= \{(x, y) \in L \times L \mid \; (x \to y) \wedge e \in H \text{ and } (y \to x) \wedge e \in H\}.$$

With this notation at hand, we can restate (3.1) in (Jipsen *et al.*, 2002).

LEMMA 2. — *Let L be a CRL. For any $\theta \in \mathrm{Con}L$ and $x, y \in L$, we have that $x\theta y$ if and only if $[(x \to y) \wedge (y \to x) \wedge e]\theta e$.*

As a consequence of this lemma we have the following result (Theorem 2.3 in (Hart *et al.*, 2002)).

LEMMA 3. — *If L is a CRL, then $\mathrm{Con}L$ is order isomorphic to $\mathrm{Sub}_C L$. The isomorphism is established through the assignments $\theta \mapsto \theta(e)$ and $H \mapsto \theta_H$.*

Recall that the *negative cone* of a CRL L is the set $L^- = \{x \in L \mid x \leq e\}$ and consider the next result (Corollary 2.8 in (Hart *et al.*, 2002)).

LEMMA 4. — *Let L be a CRL. If $a \in L^-$ then*

$$CS_L[a] = \{x \in L \mid \exists n \quad a^n \leq x \leq a^n \to e\}.$$

REMARK 5. — Observe that, if $x_1, x_2, ..., x_k$ are in the negative cone L^-, then we have that $x_1 \cdot x_2 \cdot ... \cdot x_k \leq x_1 \wedge x_2 \wedge ... \wedge x_k$. In particular, for $x \in L^-$, we can conclude that $x^n \leq x^{n-1} \leq ... \leq x^2 \leq x$. □

3. Compatible functions

DEFINITION 6. — *Let L be a CRL and let $f : L^n \to L$ be a function (not necessarily an homomorphism).*

1) We say that f is compatible with a congruence θ of L *if $(x_i, y_i) \in \theta$ for $i = 1, \ldots, n$ implies $(f(x_1, \ldots, x_n), f(y_1, ..., y_n)) \in \theta$.*

2) We say that f is a compatible function *of L provided it is compatible with all the congruences of L (Caicedo* et al.*, 2001).*

Let L be a CRL and $x \in L$. Write $x^- := x \wedge e$ and take

$$\mathbf{s}(x,y) := (x \to y)^- \cdot (y \to x)^- \tag{1}$$

LEMMA 7. — *For any $\theta \in \mathrm{Con}L$, and $x, y \in L$,*

$$x\theta y \Leftrightarrow \mathbf{s}(x,y) \in \theta(e)$$

PROOF. — Suppose that $x\theta y$. Then $(x \to y)^-$ and $(y \to x)^-$ belong to $\theta(e)$. In consequence, $\mathbf{s}(x,y) \in \theta(e)$. On the other hand, suppose that $\mathbf{s}(x,y) \in \theta(e)$; then $(x \to y)^- \cdot (y \to x)^- \theta e$. Since

$$(x \to y)^- \cdot (y \to x)^- \leq (x \to y) \wedge (y \to x) \wedge e \leq e \tag{2}$$

and $\theta(e)$ is convex, it follows, by Lemma 2, that $x\theta y$. ■

We can give a general characterization of a compatible function $f : L^k \to L$ for any CRL L as follows:

THEOREM 8. — *Let L be a CRL and let $f : L^k \to L$ be a function. Then f is compatible if and only if for every $\bar{x} = (x_1, x_2, ..., x_k), \bar{y} = (y_1, y_2, ..., y_k) \in L^k$ there exists a positive integer n such that*

$$\mathbf{s}(x_1,y_1)^n \cdot \mathbf{s}(x_2,y_2)^n \cdot ... \cdot \mathbf{s}(x_k,y_k)^n \leq \mathbf{s}(f(\bar{x}), f(\bar{y})) \tag{3}$$

PROOF. — Suppose f compatible, $\bar{x}, \bar{y} \in L^k$. We have, for $i = 1, ..., k$: $x_i\theta_i y_i$, being θ_i the principal congruence generated by (x_i, y_i). So, $x_i\Theta y_i$, for $\Theta = \bigvee \theta_i$, from where $\mathbf{s}(f(\bar{x}), f(\bar{y})) \in \Theta(e)$. But $\Theta(e) = \bigvee \theta_i(e) = \bigvee CS[\mathbf{s}_i]$, being $\mathbf{s}_i = \mathbf{s}(x_i, y_i)$ and we deduce from Theorem 2.9 in (Hart *et al.*, 2002) that $\Theta(e) = CS(\mathbf{s}_1 \wedge ... \wedge \mathbf{s}_k)$. Then, there exist positive integers $n_1, ..., n_k$ such that $\mathbf{s}_1^{n_1} \cdot ... \cdot \mathbf{s}_k^{n_k} \leq \mathbf{s}(f(\bar{x}), f(\bar{y}))$. Take $n = \max\{n_1, ..., n_k\}$.

Conversely, suppose that (3) holds and let be, $x_i\phi y_i$, for $i = 1, ..., k$, for a given congruence ϕ. Then, $\mathbf{s}_i \in \phi(e)$ for every $i = 1, ..., k$, from where for every n, $\mathbf{s}_1^n \cdot ... \cdot \mathbf{s}_k^n \in \phi(e)$. Since the image of $\mathbf{s}$ is included in the negative cone of L, we are done. ■

Some particular cases of Theorem 8 are well known results (see (Caicedo *et al.*, 2001) and (Ertola *et al.*, 2007)). We shall now describe more in detail some other important cases for one-variable functions.

3.1. BL-algebras

Recall that a residuated lattice $(A, \wedge, \vee, \cdot, \to, 0, 1)$ is a *BL-algebra* if and only if the following two identities hold for all $x, y \in A$:

1) $x \wedge y = x \cdot (x \rightarrow y)$,
2) $(x \rightarrow y) \vee (y \rightarrow x) = 1$

COROLLARY 9. — *Let A be a BL-algebra. Then, for any function $f : A \rightarrow A$ the following conditions are equivalent:*

1) For every x, y of A there exists an n such that $(\mathbf{s}(x,y))^n \leq \mathbf{s}(fx, fy)$
2) f is compatible.

3.2. *ℓ-groups*

A system $(G, \wedge, \vee, +, -, 0)$ is a *lattice-ordered group* or *ℓ-group* if $(G, \wedge, \vee)$ is a distributive lattice and $(G, +, -, 0)$ is a commutative group such that

$$(a \vee b) + t = (a + t) \vee (b + t)$$

holds in G.

Let x be an element of an ℓ-group G. Set $x^+ = x \vee 0$ and $|x| = x^+ + (-x)^+$, $n|x| = |x| + |x| + ... + |x|$ (n times). Let $T(x,y) = |x - y|$.

COROLLARY 10. — *Let G be an ℓ-group and let $f : G \longrightarrow G$ be a function. Then the following statements are equivalent.*

1) For every $x, y \in A$ there is an n such that $T(fx, fy) \leq n(T(x,y))$
2) f is compatible.

As is customary for ordered groups, Corollary 10 is stated in terms of the positive cone instead of the negative one. Since $\mathbf{s}$ and T are related by $\mathbf{s}(x,y) = -T(-x,-y)$, this result is just another corollary of Theorem 8.

REMARK 11. — Let L be a CRL and $x, y \in L$. Take

$$\mathbf{t}(x,y) := [(x \rightarrow y) \wedge (y \rightarrow x)] \wedge e = (x \leftrightarrow y)^- \qquad (4)$$

It is easy to see that $CS(\mathbf{t}(x,y)) = CS(\mathbf{s}(x,y))$. Hence all the results of this section also hold if we replace $\mathbf{s}$ by $\mathbf{t}$. $\square$

3.3. *Local affine completeness*

We finish this section applying Theorem 8 to the proof of the locally affine completeness of the category of CRLs.

THEOREM 12. — *Let $f : L^k \rightarrow L$ be a compatible function, B be a finite subset of L^k and $\bar{x} \in B$. If we let $T_{\bar{x}} = \{\mathbf{s}(b_1, x_1)^{n_b} \cdot ... \cdot \mathbf{s}(b_k, x_k)^{n_b} \cdot f(\bar{b}) \mid \bar{b} \in B\}$, where $n_b := \max\{n_{(\bar{b},\bar{x})} \mid \bar{x} \in B\}$ and $n_{(\bar{b},\bar{x})}$ is the integer associated to the pair $(\bar{b}, \bar{x})$ in Theorem 8. Then, $f(\bar{x}) = \bigvee T_{\bar{x}}$.*

PROOF. — Let $\bar{x} \in B$. For every $\bar{b} \in B$ we have that

$$\mathbf{s}(b_1, x_1)^{n_b} \cdot ... \cdot \mathbf{s}(b_k, x_k)^{n_b} \leq \mathbf{s}(f(\bar{b}), f(\bar{x})) =$$

$$(f(\bar{b}) \to f(\bar{x}))^- \cdot (f(\bar{x}) \to f(\bar{b}))^- \leq (f(\bar{b}) \to f(\bar{x}))^- \leq f(\bar{b}) \to f(\bar{x})$$

Then,

$$\mathbf{s}(b_1, x_1)^{n_b} \cdot ... \cdot \mathbf{s}(b_k, x_k)^{n_b} \cdot f(\bar{b}) \leq f(\bar{x}),$$

which proves that $f(\bar{x})$ is an upper bound of $T_{\bar{x}}$. On the other hand, since $\mathbf{s}(x_i, x_i) = e$, we have that $\mathbf{s}(x_1, x_1)^{n_b} \cdot ... \cdot \mathbf{s}(x_k, x_k)^{n_b} \cdot f(\bar{x}) = f(\bar{x}) \in T_{\bar{x}}$, from where the result follows. ■

COROLLARY 13. — *Every CRL is locally affine complete.*

4. Pre-compatible functions

The notion of succesor is a very natural one and has been treated since the axiomatization of positive integers by Peano. In Heyting chains there is a generalization of that notion arising from the fact that, for $x \neq 1$,
(*)"$y \to x$ less or equal x" is equivalent to "$x < y$" and so, if the set of those y fulfilling (*) has a minimum $S(x)$, this minimum will be the succesor of x. It is clear that in any Heyting algebra there exists $S(x)$ for every x if and only if the function $S(x)$ satisfies (S1) and (S2):

(S1) $S(x) \to x \leq x$,

(S2) $S(x) \leq y \vee (y \to x)$.

From a logical point of view, it is possible to define implicitly a connective S in intuitionistic propositional calculus (IPC, for short) by the following axioms:

(1) $(S(p) \to p) \to p$,

(2) $S(p) \to (q \vee (q \to p))$.

This connective plays a very important role in some logics. Fixed a positive integer n, let $\mathcal{L}_n$ be the the logic obtained by adding to the IPC the following axioms:

(L1) $(p \to q) \vee (q \to p)$,

(L2) $(p_1 \to p_2) \vee ... \vee (p_n \to p_{n+1})$.

Then, every implicit connective of $\mathcal{L}_n$ is generated by S (Caicedo *et al.*, 2001).

It is an open problem if this is so for entire IPC.

In this section we extend the definition of successor to CRLs and, furthermore, we explore other functions different from $y \to x$ that allow the definition of compatible functions on CRLs in an analogous way.

DEFINITION 14. — *Let L be a poset. A function $f : L \times L \to L$ is called* pre-compatible *if the following condition holds:*

(PRE) for every $a \in L$, the set $\{y \mid f(a, y) \leq y\}$ has a least element.

We are not assuming that f is monotone.

For each pre-compatible function $f : L \times L \to L$ we denote by $\overline{f} : L \to L$ the function that to each x in L assigns the least element of $\{y \mid f(x, y) \leq y\}$.

LEMMA 15. — *Let $f : L \times L \to L$ be a binary function on a partial order with binary suprema $(L, \vee)$ such that:*

(M) for all $a, b, c \in L$, $c \geq b$ implies $f(a, c) \leq f(a, b)$.

Then, the following are equivalent.

1) f is pre-compatible
2) there exists a function $g : L \to L$ such that

(S1) $f(x, gx) \leq gx$

(S2) $gx \leq y \vee f(x, y)$.

Moreover, in this case, $g = \overline{f}$.

PROOF. — If f is pre-compatible and satisfies (M), then it is not difficult to check that $\overline{f}$ (in the place of g) satisfies conditions (S1) and (S2). Conversely, assume that g satisfies the stated equations. Condition (S1) says that gx is in the set $\{y \mid f(x, y) \leq y\}$. On the other hand, let y be such that $f(x, y) \leq y$. Then condition (S2) implies that $gx \leq y \vee f(x, y) \leq y \vee y = y$. Then gx is the least element of $\{y \mid f(x, y) \leq y\}$. That is, $g = \overline{f}$. ■

THEOREM 16. — *Let L be a CRL and $f : L \times L \to L$ be a pre-compatible function. If, for every $x, y \in L$ there exists a positive integer n such that:*

$$f(x, \overline{f}y) \cdot ((x \to y)^-)^n \leq f(y, \overline{f}y) \tag{5}$$

then $\overline{f}$ is compatible.

PROOF. — Suppose that (5) holds in L. We have, by (S2), that

$$\overline{f}x \leq \overline{f}y \vee f(x, \overline{f}y)$$

and so,

$$(\overline{f}x) \cdot ((x \to y)^-)^n \leq ((\overline{f}y) \vee f(x, \overline{f}y)) \cdot ((x \to y)^-)^n$$

from where we can conclude that

$$(\overline{f}x) \cdot ((x \to y)^-)^n \leq [(\overline{f}y) \cdot ((x \to y)^-)^n] \vee [f(x, \overline{f}y) \cdot ((x \to y)^-)^n]$$

because $\cdot$ distributes over $\vee$. Now $(\overline{f}y) \cdot ((x \to y)^-)^n \leq (\overline{f}y)$ and

$$f(x, \overline{f}y) \cdot ((x \to y)^-)^n \leq f(y, \overline{f}y)$$

by hypothesis. So that

$$(\overline{f}x)\cdot((x\to y)^-)^n \leq (\overline{f}y)\vee f(y,\overline{f}y)=\overline{f}y$$

by (S1) and then $((x\to y)^-)^n \leq (\overline{f}x)\to(\overline{f}y)$ by adjointness, and also:

$$((x\to y)^-)^n \leq ((\overline{f}x)\to(\overline{f}y))^-.$$

In the same way:

$$((y\to x)^-)^n \leq ((\overline{f}y)\to(\overline{f}x))^-.$$

So

$$((x\to y)^-)^n\cdot((y\to x)^-)^n \leq ((\overline{f}x)\to(\overline{f}y))^-\cdot((\overline{f}y)\to(\overline{f}x))^-,$$

that is

$$(\mathbf{s}(x,y))^n \leq \mathbf{s}(\overline{f}x,\overline{f}y)$$

Compatibility follows from Theorem 8. ■

5. Variations on the successor

The term $y\to x$ induces a binary function $f(x,y)$ on every CRL. When this function is pre-compatible we denote $\overline{f}$ by S and call it the *successor*. In this case, Theorem 16 is applicable and so S is compatible.

For example, a *generalized Heyting algebra* (see (Monteiro, 1985)) is an integral commutative residuated lattice $(H,\wedge,\vee,\cdot,\to,1)$ such that $\cdot=\wedge$. Heyting algebras are exactly generalized Heyting algebras with a minimum. The successor operation in Heyting algebras are studied in (Caicedo *et al.*, 2001). The case of generalized Heyting algebras is treated in (Ertola *et al.*, 2007).

EXAMPLE 17. — Consider the MV-algebra $[0,1]$ with its well known structure. It is easy to prove that for $t\in[0,1]$, $\{y \mid y\to t\leq y\}$ is the interval $[\frac{1+t}{2},1]$. Then $S(t)=\frac{1+t}{2}$. □

EXAMPLE 18. — Let $\mathbb{Z}$ be the totally ordered ℓ-group of the integers. Then, for any $n\in\mathbb{Z}$, we have that

$$S(n)=\min\{m \mid m\to n\leq m\}=\min\{m \mid (n-m)\leq m\}$$

and $n-m\leq m$ iff $n\leq m+m=2m$. Then, we conclude that $S(n)=\frac{n}{2}$ when n is even and that $S(n)=\frac{n+1}{2}$ when n is odd. □

For each $n\geq 0$ the term $P_n(x,y)=y^n\to x$ induces a binary function on any CRL L. When this function is pre-compatible Theorem 3 is applicable, just as in the case of the successor, and we denote the resulting compatible function by $S_n : L\to L$. Of course, $S_1=S$.

EXAMPLE 19. — Take $L = ([0,1], \wedge, \vee, \cdot, \to, 1)$ the product algebra (Hájek, 1998); *i.e.*, the real interval $[0,1]$ with its usual order and product. We have that

$$a \to b = \begin{cases} 1, \text{ if } a \leq b \\ b/a, \text{ otherwise} \end{cases}$$

and hence, for $x \neq 0$, we have that

$$y^n \to x = x/y^n \leq y \quad \text{iff} \quad x \leq y^{n+1}$$

Thus, $S_n(x) = \sqrt[n+1]{x}$, for every $x \in (0,1]$. □

5.1. *Affine completeness of enriched CRLs*

It is well known that the category of CRLs is not affine complete (see (Kaarli *et al.*, 2001)). On the other hand, it is not known whether the category of Heyting algebras equipped with successor is affine complete. We can ask the same question but replacing Heyting algebras with CRLs. In this section we prove that the answer is negative. Indeed, we give an example of a CRL with both S_1 and S_2 but where S_2 is not definable in terms of S_1.

Having done this, we found it natural to ask ourselves if the category of CRLs with S_2 is affine complete. We show that the answer is negative again. The main instrument in both proofs is the following structure.

EXAMPLE 20. — Let $\mathbf{Q}$ be the ordered abelian group $(\mathbb{Q}, \vee, \wedge, +, -, 0)$. For this structure, $S_1(r) = \frac{r}{2}$ and $S_2(r) = \frac{r}{3}$. □

Denote the structure $(\mathbb{Q}, \vee, \wedge, +, -, S_1, 0)$ by $\mathbf{Q}_1$ and $(\mathbb{Q}, \vee, \wedge, +, -, S_2, 0)$ by $\mathbf{Q}_2$. Observe that $(\mathbb{Q}, \vee, \wedge)$ is distributive, that $+$ distributes over $\vee$ and $\wedge$ and that division distributes over $\vee, \wedge$ and $+$. These facts will play an important role in the main result of this section so let us state the key consequence.

LEMMA 21. — *Every polynomial $P(x)$ in the structure $\mathbf{Q}_1$ is equivalent (as a function $\mathbb{Q} \to \mathbb{Q}$) to one of the form*

$$(p_{11} \wedge p_{12} \wedge \ldots \wedge p_{1n_1}) \vee \ldots \vee (p_{m1} \wedge p_{m2} \wedge \ldots \wedge p_{mn_m})$$

with $p_{ij}x = a_{ij} + b_{ij}\frac{x}{2^{k_{ij}}}$ and $a_{ij} \in \mathbb{Q}$, $b_{ij} \in \mathbb{Z}$ and $k_{ij} \in \mathbb{N}$ for every i and j.

PROOF. — A straightforward induction. ■

We can now state one of the main results of this section.

PROPOSITION 22. — *The structure $\mathbf{Q}_1$ is not affine complete.*

PROOF. — We show that S_2 cannot be given by a polynomial. For let $P(x)$ be a polynomial in $\mathbf{Q}_1$. By Lemma 21, for every $t \in \mathbb{Q}$, $Pt = a_{ij} + b_{ij}\frac{t}{2^{k_{ij}}}$ for some $1 \leq i, j \leq n$. Then $S_2(t) = P(t)$ if and only if $\frac{t}{3} = a_{ij} + b_{ij}\frac{t}{2^{k_{ij}}}$ if and only if

$(2^{k_{ij}} - 3b_{ij})t = 3(2^{k_{ij}})a_{ij}$. Notice that $2^{k_{ij}} - 3b_{ij}$ cannot be 0. For if it was then we would have $b_{ij} = \frac{2^{k_{ij}}}{3}$ which is not an integer (contradicting Lemma 21) since $(2,3) = 1$. We then have that $t = \frac{3(2^{k_{ij}})a_{ij}}{2^{k_{ij}} - 3b_{ij}}$. Hence, $P(t) = S_2(t)$ has only finite solutions. So S_2 cannot be a polynomial in $\mathbf{Q}_1$. ■

As a corollary one obtains that the variety of CRLs enriched with S_1 is not affine complete. But notice that since $\mathbf{Q}_1$ has a lot more structure than 'just' that of a CRL, we can also conclude that the variety of l-groups enriched with S is not affine complete. There are also variations of these corollaries obtained by considering successor functions satisfying, for example, preservation of meets or sups.

The second result uses essentially the same idea.

PROPOSITION 23. — *The structure $\mathbf{Q}_2$ is not affine complete.*

PROOF. — First prove a result analogous to Lemma 21 but which says that every polynomial $P(x)$ in the structure $\mathbf{Q}_2$ is equivalent to one of the form

$$(p_{11} \wedge p_{12} \wedge \ldots \wedge p_{1n_1}) \vee \ldots \vee (p_{m1} \wedge p_{m2} \wedge \ldots \wedge p_{mn_m})$$

with $p_{ij}x = a_{ij} + b_{ij}\frac{x}{3^{k_{ij}}}$ for every i and j. Then use the same argument as in Proposition 22 to prove that S_1 cannot be given by a polynomial. ■

It follows that the variety of CRLs enriched with S_2 is not affine complete.

5.2. *CRLs with one operation but without the other*

The first two examples present CRLs where S_1 is defined but where S_2 is not.

EXAMPLE 24. — Let $X = \{r \in \mathbb{Z} \mid r = \frac{p}{q}, (p,q) = 1 \text{ and } 3 \nmid q\}$. It is clear that X is an ordered subgroup of $(\mathbb{Z}, +, 0)$. In our present case $S_1(x) = min\{y \mid x \leq 2y\}$ and $S_2(x) = min\{y \mid x \leq 3y\}$. Hence $S_1(x) = \frac{x}{2}$ is well-defined in X but, for example $\{y \mid 1 \leq 3y\}$ has no minimum in X. So $S_2(1)$ is not defined. □

For the second example let $\mathbb{E}$ be the set of complex numbers constructible with straightedge and compass ((Hungerford, 1974), Ch.V).

EXAMPLE 25. — Take $A = (0,1] \cap \mathbb{E}$ with its inherited operations. Since each element of A has a square root in A, it follows that the successor $S_1 x = \sqrt{x}$ is defined for every $x \in A$. On the other hand, consider $1/2 \in A$. $\sqrt[3]{1/2}$ is a root of the irreducible polynomial $2x^3 - 1 \in \mathbb{Q}(i)[x]$, and in consequence its characteristic polynomial is of degree 3, which is not a power of 2. Thus, $\sqrt[3]{1/2} \notin A$. Consider now the set $E_2(1/2) = \{y \in A \mid 1/2 \leq y\}$. Since $\mathbb{Q} \subseteq \mathbb{E}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, A is dense in $(0,1]$. Thus, $E_2(1/2)$ has no minimum. So we have proved that S_2 cannot be defined on A. □

The next example presents a CRL with S_2 but without S_1.

EXAMPLE 26. — Consider the ordered abelian group (and hence CRL) $(\mathbb{Q}, +, 0)$. Write $\mathbb{P} := \{p/2q \in \mathbb{Q} \mid p, q \in \mathbb{Z} - \{0\},\ (p,q) = (p,2) = 1\}$. It is clear that $1 \notin \mathbb{P}$. Furthermore, if $x, y \notin \mathbb{P}$, then there are nonzero integers m, n, r and s, such that $x = m/n$, $y = r/s$, $(m,n) = (r,s) = 1$ and n and s are odd. Thus,

$$x + y = \frac{m}{n} + \frac{r}{s} = \frac{ms + rn}{ns}$$

Since the denominator can never be even we have that $x + y \notin \mathbb{P}$. So we have just shown that $\mathbb{S} := \mathbb{Q} - \mathbb{P}$ is an ordered subgroup of $\mathbb{Q}$. Let us now see what happens with the successor and S_2 in $\mathbb{S}$. Take $E_1(1) = \{y \in \mathbb{S} \mid 1 \leq 2y\}$. If we consider the sequence $\{\frac{n}{2n-1} \mid n \geq 1\}$ which is contained in $\mathbb{S}$ and converges from above to $1/2$, we conclude that $E_1(1)$ does not have a minimum in $\mathbb{S}$. Hence we have no successor on $\mathbb{S}$. On the other hand, for every $x \in \mathbb{S}$, $E_2(x) = \{y \in \mathbb{S} \mid x \leq 3y\}$ has always a minimum in $\mathbb{S}$: $S_2(x) = x/3$. □

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