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On Some Categories of Involutive Centered Residuated Lattices

Abstract. Motivated by an old construction due to J. Kalman that relates distributive lattices and centered Kleene algebras we define the functor $\mathsf{K}^\bullet$ relating integral residuated lattices with 0 (IRL_0) with certain involutive residuated lattices. Our work is also based on the results obtained by Cignoli about an adjunction between Heyting and Nelson algebras, which is an enrichment of the basic adjunction between lattices and Kleene algebras. The lifting of the functor to the category of residuated lattices leads us to study other adjunctions and equivalences. For example, we treat the functor C whose domain is cuRL, the category of involutive residuated lattices M whose unit is fixed by the involution and has a Boolean complement $\mathbf{c}$ (the underlying set of $\mathsf{C}M$ is the set of elements greater or equal than $\mathbf{c}$). If we restrict to the full subcategory NRL of cuRL of those objects that have a nilpotent c, then C is an equivalence. In fact, $\mathsf{C}M$ is isomorphic to $\mathsf{C_e}M$, and $\mathsf{C_e}$ is adjoint to $\widehat{(\_)}$, where $\widehat{(\_)}$ assigns to an object A of IRL_0 the product $A \times A^0$ which is an object of NRL.

Keywords: residuated lattices, involution, Kalman functor.

1. Introduction

Much of the present work is motivated by results due to Kalman relating lattices and Kleene algebras [8] and the conviction that these can be lifted to the level of residuated lattices.

In [3], Cignoli shows that one of the key constructions in [8] induces a functor from the category of bounded distributive lattices into the category of Kleene algebras and that it has a left adjoint. Cignoli also shows that this adjunction restricts to one between the categories of Heyting algebras and of Nelson algebras and observes that this restriction is essentially a result obtained independently by Vakarelov [10] and Fidel [4]. Moreover, it is proved in [3], that Kalman's adjunction can be further restricted to an adjunction between the category of Boolean algebras and that of three valued Lukasiewicz algebras and also to an adjunction between the category of Stone algebras and that of regular α-De Morgan algebras. Cignoli uses the properties of the Kalman adjunction in order to provide natural proofs

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of results involving these categories of algebras. Of particular relevance is the fact that the left adjoints to the diferent restrictions of the functor K preserve lattices of congruences.

These applications of Kalman's construction to different varieties of Heyting algebras, together with the recent interest on residuated lattices, suggest that it is potentially fruitful to understand Kalman's work in the context of residuated lattices. We do this in the present paper.

Our work will involve different functors into the category of residuated lattices with involution. Some of the more interesting results involve the discovery of 'good' subcategories of residuated lattices with involution. Our constructions will factor through these subcategories and the resulting factorizations will have better properties than the original ones.

In many cases, the process of finding such subcategories was inspired by a result at the level of distributive lattices and Kleene algebras. So we recall this in Section 3.

In Sections 2 and 3 we present a new approach to the old construction due to Kalman. In fact, we show that given a distributive lattice A the functor M, that assigns to A the De Morgan algebra $A^0 \bigoplus A$ can be factored through the category of centered Kleene algebras. Moreover, M has a right adjoint L, which has a further adjoint K. This one is Kalman functor. When K is restricted to Heyting algebras and L to the Nelson algebras the adjunction $\mathsf{L} \dashv \mathsf{K}$ becomes an equivalence, as Cignoli proved in [3]. This fact (mentioned in 3.1) motivates the lifting of the construction to residuated lattices in general, as we remark in 3.2. In the original setting we begin with a functor $\widehat{(\_)}$ that goes from distributive lattices to De Morgan algebras. In section 4 we lift $\widehat{(\_)}$ to a functor from the category IRL_0 (integral residuated lattices with 0) to iRL_b (involutive residuated bounded lattices).

We introduce *cones* in section 5. The objects of the category $\mathbf{c}$iRL are those of iRL that have a distinguished element that is a "center", i.e., is fixed for the involution. We call $\mathbf{e}$iRL the full subcategory of $\mathbf{c}$iRL determined by the residuated lattices whose center is precisely the identity $\mathbf{e}$.

An interesting subcategory of $\mathbf{c}\mathrm{iRL}_\mathrm{b}$ is PRL (proto-integral residuated lattices). For an algebra A of $\mathbf{c}\mathrm{iRL}_\mathrm{b}$ we ask for the following additional condition: the greatest element $\mathbf{1}$ behaves as neutral for elements greater than the center $\mathbf{c}$.

A *complemented-unit bounded involutive residuated lattice* is an object of $\mathbf{e}$iRL with two distinguished elements $\mathbf{e}$ and $\mathbf{c}$, being $\mathbf{e}$ the unit and $\mathbf{c}$ its Boolean complement. We call this category cuRL. If the condition $\mathbf{c} * \mathbf{c} = 0$ holds in an algebra A we say that A is a *nil-complement-unit* *residuated lattice.* We prove (Lemma 5.6) that the functor $\widehat{(\_)}$ factors through $\mathrm{NRL} \longrightarrow \mathrm{IRL_b}$.

In section 6 we show that this embedding is the largest "convenient" subcategory (see Section 2) and moreover that $\widehat{(\_)} : \mathrm{IRL}_0 \longrightarrow \mathrm{NRL}$ is actually an equivalence. Also, the right adjoint associated to the largest convenient subcategory has a further right adjoint.

The original Kalman construction applied to a lattice A gives the Kleene algebra $KA = \{(x, y) \in \widehat{A} \mid x \wedge y = 0\}$. Motivated by [9] we replace "$\wedge$" by "$\cdot$" and define in section 7, for a residuated lattice A:

$$\mathsf{K}^{\bullet} A := \{(x, y) \in \widehat{A} \mid x \cdot y = 0\}$$

The assignment $\mathsf{K}^{\bullet}$ extends to a functor (see Lemma 7.1).

The unit $(1, 1)$ of $\widehat{A}$ does not belong to $\mathsf{K}^{\bullet} A$, but $\mathsf{K}^{\bullet} A$ is closed by the operations $\dot{\to}$ and $*$ and has the unit $(1, 0)$ (is integral).

A c-differential lattice is an integral involutive residuated lattice with a center c which satisfies the "Leibniz condition"(see Definition 7.2). We denote this category by DRL.

In Theorem 7.6 we prove that $\mathsf{K}^{\bullet}$ has a left adjoint C. As in Theorem 3.2, C has a further left adjoint. As a corollary (see 7.8) we prove that when condition ($\mathsf{C}\mathsf{K}^{\bullet}$) holds then the adjunction $\mathsf{C} \dashv \mathsf{K}^{\bullet}$ restricts to an equivalence. The rest of section 7 is devoted to give an interesting factorization of $\mathsf{K}^{\bullet}$ through $\widehat{(\_)}$ and to give an explicit description of the isomorphisms between congruence lattices associated to the equivalence $\mathsf{C} \dashv \mathsf{K}^{\bullet}$.

In the section 8 we define functors $M^{\bullet}$ and $M_{\bullet}$ adjoint to the cone functors C and $\mathsf{C}_{\mathbf{e}}$.

In section 9 we go back to the motivation: the adjunction that relates Heyting algebras to Nelson algebras. We show our point of view after previous results. As a particular case we consider Heyting chains.

Apendix A contains a categorical construction of some of the left adjoints considered in this paper. Appendix B shows that some of the preservation of congruences results follow from purely categorical properties of adjoint situations of Kalman type.

2. A useful categorical guide

In the process of our research on this topic we have found it useful to place our problems in the following general context.

Let $F : \mathcal{C} \to \mathcal{D}$ a functor. (In practice this will be a functor between varieties. But for the moment let us forget about that.)

Definition 2.1. We say that a subcategory $\mathcal{D}' \to \mathcal{D}$ is *convenient* (with respect to $F : \mathcal{C} \to \mathcal{D}$) if F factors through $\mathcal{D}' \to \mathcal{D}$ and moreover, the factorization $F : \mathcal{C} \to \mathcal{D}'$ has a right adjoint.

So a convenient subcategory induces a diagram as below

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, shift left, "F"] & \mathcal{D} \\ & \mathcal{D}' \arrow[u] \arrow[ul, shift left, "G"] \end{tikzcd}$$

with the triangle starting on $\mathcal{C}$ commuting and $F \dashv G$.

With a fixed functor $F : \mathcal{C} \to \mathcal{D}$ at hand, one can then ask what is the largest convenient subcategory of $\mathcal{D}$. This has proved to have interesting answers among the categories of lattices we have studied and, perhaps curiously, the way to find the answers can be explained in very simple abstract terms as follows.

If a factorization $F : \mathcal{C} \to \mathcal{D}'$ is to have a right adjoint then, letting 0 be the initial object of $\mathcal{C}$, $F0$ must be initial in $\mathcal{D}'$. We can consider the subcategory of $\mathcal{D}$ determined by those objects that have a unique map from $F0$. If this subcategory was convenient then it would clearly be the largest one. This is exactly what happens in the examples that Kalman's work has motivated us to study.

3. Distributive lattices and Kleene algebras

We assume the reader is familiar with *bounded distributive lattices.* A *De Morgan algebra* is an algebra $(A, \vee, \wedge, \sim, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\sim$ fulfills the equations

M1 $\sim\sim x = x$

M2 $\sim (x \vee y) = \sim x \wedge \sim y, \quad \sim (x \wedge y) = \sim x \vee \sim y.$

We denote by DL and DM the categories of distributive lattices and of De Morgan algebras respectively. Now let X be a distributive lattice X, the poset $X \times X^{op}$ is a lattice. Its meet and join operations can be given explicitly by $(x, x') \vee (y, y') = (x \vee y, x' \wedge y')$ and $(x, x') \wedge (y, y') = (x \wedge y, x' \vee y')$ and the resulting lattice is distributive. Moreover, it is a De Morgan algebra when equipped with $\sim (x, x') = (x', x)$. In this way we obtain a functor $\widehat{(\_)} : \mathrm{DL} \to \mathrm{DM}$. This functor will play a very important role in later sections.

We will also need to consider the subalgebra of $\widehat{X}$ determined by the set $\mathsf{M}X = \{(x, x') \in X \times X \mid x = 0 \textit{ or } x' = 0\}$. It is easy to see that this set is closed under the operations $\vee$, $\wedge$ and $\sim$. The following is then not difficult to prove.

LEMMA 3.1. *For any distributive lattice X, $\mathsf{M}X$ is a De Morgan algebra. Moreover, the assignment $X \mapsto \mathsf{M}X$ can be extended to a functor $\mathsf{M} : \mathrm{DL} \to \mathrm{DM}$.*

It is useful to picture $\mathsf{M}X$ as the lattice X together with an up-side-down copy of itself glued together through the bottom element of X.

THEOREM 3.2. *There is a largest convenient subcategory $\mathcal{E}$ of* DM *with respect to $\mathsf{M} : \mathrm{DL} \to \mathrm{DM}$. Moreover:*

1. *$\mathcal{E}$ is the category of algebras for a finitary algebraic theory. Indeed, it can be described as the category Kl_c of centered Kleene algebras,*
2. *the right adjoint $\mathsf{L} : \mathrm{Kl}_c \to \mathrm{DL}$ is faithful and*
3. *it has a further right adjoint $\mathsf{K} : \mathrm{DL} \to \mathcal{E}$.*

In order to prove Theorem 3.2 we will give an explicit definition of $\mathcal{E}$.

DEFINITION 3.3. A *Kleene algebra* is a De Morgan algebra in which the axiom

(K) $x \wedge \sim x \leq y \vee \sim y$.

holds.

A Kleene algebra is called *centered* if it has a center. That is, an element c such that $\sim c = c$. We denote the category of centered Kleene algebras by Kl_c.

Following the suggestion made at the end of Section 2 we can start by looking at the De Morgan algebras that have a unique morphism from $\mathsf{M}0$. But $\mathsf{M}0$ is the total order with three elements. So, for a De Morgan algebra X, there exists a unique $\mathsf{M}0 \to X$ if and only X has a unique center. Theorem 3.2 would be then proved if we could show that the category of De Morgan algebras with a unique center satisfies the properties in the statement.

PROPOSITION 3.4 (See [8]). *If A is a De Morgan algebra then the following are equivalent:*

1. *A is a centered Kleene algebra,*
2. *A has a unique center,*
3. *there exists a center c such that for every $x \in A$, $x \leq\sim x$ implies $x \leq c$.*

PROOF. The first item trivially implies the second. To prove that the second implies the third, assume that A has a unique center c and suppose that there exists an $x \in A$ such that $x \leq\sim x$. Now, let $d = (c\wedge \sim x) \vee x$. Then we have

$$\sim d =\sim (c\wedge \sim x)\wedge \sim x = (c \vee x)\wedge \sim x =$$
$$= (c\wedge \sim x) \vee (x\wedge \sim x) = (c\wedge \sim x) \vee x = d$$

so that d is a center. Then $d = c$ and in particular $(c\wedge \sim x) \vee x \leq c$. But this implies that $x \leq c$.

Finally, to prove that the third item implies the first, let $a =\sim x \wedge x$ and $b =\sim y \vee y$. Then $a \leq\sim a$ and $\sim b \leq\sim\sim b = b$. By hypothesis, $a \leq c \leq b$. Then, (K) holds. ■

Morphisms of De Morgan algebras trivially preserve centers so it is clear that the full subcategory of DM determined by those which have a unique center is exactly the category of Kl_c of centered Kleene algebras. Now, Proposition 3.4 implies that $\mathsf{M} : \mathrm{DL} \to \mathrm{DM}$ factors through the category Kl_c of centered Kleene algebras. We denote the factorization by $\mathsf{M} : \mathrm{DL} \to \mathrm{Kl}_c$.

LEMMA 3.5. *The functor $\mathsf{M} : \mathrm{DL} \to \mathrm{Kl}_c$ has a right adjoint denoted by $\mathsf{L} : \mathrm{Kl}_c \to \mathrm{DL}$ which assigns to each centered Kleene algebra A with center c, the lattice $\mathsf{L}A$ determined by $\{a \in A \mid c \leq a\}$.*

PROOF. Consider the natural transformation $\eta : Id \to \mathsf{LM}$ given by the obvious iso. Now let $f : A \to \mathsf{L}X$ be a morphism of distributive lattices. It is straightforward to check that the extension $\overline{f} : \mathsf{M}A \to X$ given by $\overline{f}(x, 0) = fx$ and $f(0, y) =\sim (fy)$ is a morphism of De Morgan algebras (and so of Kleene algebras). Moreover, it is the unique one such that $\overline{f}\eta = f$. So the result follows. ■

To finish the proof of Theorem 3.2 we have to prove that $\mathsf{L} : \mathrm{Kl}_c \to \mathrm{DL}$ has a right adjoint. But this is Cignoli's result. For each bounded distributive lattice D, define the centered Kleene algebra $\mathsf{K}D$ with underlying set $\{(x, y) \in D \times D \mid x \wedge y = 0\}$ and lattice operations inherited from $\widehat{D}$.

THEOREM 3.6. (See Theorem 1.7 in [3].) *For each bounded distributive lattice D, $\mathsf{K}D$ is a centered Kleene algebra (with center $(0, 0)$). Moreover, the construction extends to a functor $\mathsf{K} : \mathrm{DL} \to \mathrm{Kl}_c$ that is right adjoint to $\mathsf{L} : \mathrm{Kl}_c \to \mathrm{DL}$.*

The proof of Theorem 3.2 is finished.

3.1. Lifting the adjunction $\mathsf{L} \dashv \mathsf{K} : \mathrm{DL} \to \mathrm{Kl}_c$

It is clear from [3] that some of the results loc. cit. are applications of the fact that the adjunction $\mathsf{L} \dashv \mathsf{K} : \mathrm{DL} \to \mathrm{Kl}_c$ lifts to other varieties. For example, consider the case of Heyting algebras. (We assume that the reader is familiar with Heyting algebras and we denote the category of these by Hey.)

A *quasi-Nelson algebra* is a Kleene algebra A such that for each $a, b \in A$, the relative pseudocomplement $a \Rightarrow (\sim a \vee b) = a \to b$ exists. It is well-known (see [2]) that the class of quasi-Nelson algebras is a variety of algebras $(A, \vee, \wedge, \to, \sim, 0, 1)$ of type $(2, 2, 2, 1, 0, 0)$. A quasi-Nelson algebra satisfying

(N) $(a \wedge b) \to c = a \to (b \to c)$

is called a *Nelson algebra*. We denote the category of centered Nelson algebras by Nel_c.

Proposition 3.7 (Theorem 3.14(ii) in [3]). *Kalman's adjunction restricts to an adjoint equivalence* $\mathsf{L} \dashv \mathsf{K} : \mathrm{Hey} \to \mathrm{Nel}_c$.

Further examples of liftings of the adjunction $\mathsf{L} \dashv \mathsf{K} : \mathrm{DL} \to \mathrm{Kl}_c$ are given by Boolean algebras vs Three-valued Lukasiewicz algebras and Stone algebras vs regular α-De Morgan algebras but we refer to [3] for these.

3.2. Lifting to the case of residuated lattices

The fact that Kalman's construction can be extended consistently to Heyting algebras led us to believe that some of this picture could be lifted to residuated lattices in general. The rest of the paper deals with this level of generality.

Let IRL_0 be the category of integral residuated distributive lattices with initial object 0. The first step into our investigations is to isolate a reasonable category which completes the following diagram

$$\begin{tikzcd} \mathrm{IRL}_0 \arrow[r] \arrow[d] & ? \arrow[d] \\ \mathrm{DL} \arrow[r, "\widehat{(\_)}"'] & \mathrm{DM} \end{tikzcd}$$

where $\mathrm{IRL}_0 \to \mathrm{DL}$ is the obvious forgetful functor and $\widehat{(\_)} : \mathrm{DL} \to \mathrm{DM}$ is the functor introduced at the beginning of Section 3. The category completing this square should be a De Morgan algebra equipped with a residuated lattice structure in such a way that the two structures interact in a reasonable way. In Section 4 we introduce such a category.

4. Involutive residuated lattices

In this section we shall recall those definitions and properties of involutive residuated lattices we shall use in this work. For a more detailed exposition on this subject we refer the reader to [5] and [9].

An *involutive residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \sim, e\rangle$ such that

1. $(L, \wedge, \vee)$ is a lattice
2. $(L, \cdot, e)$ is a monoid
3. $\sim$ is an involution of the lattice that is a dual automorphism; i.e., $\sim(\sim x) = x$ for all $x \in L$, and
4. $x \cdot y \leq z$ iff $y \leq \sim((\sim z) \cdot\ x)$ iff $x \leq \sim(y \cdot (\sim z))$.

We shall write $x\backslash y = \sim((\sim y) \cdot x)$ and $y/x = \sim(x \cdot (\sim y))$. We shall write iRL for the category of involutive residuated lattices. There are two term-equivalent descriptions of iRL. An algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ is said to be a *dualizing residuated* lattice provided it satisfies the following conditions:

1. $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice (RL); and
2. d is a cyclic dualizing element. That is, for all $x \in L$,
   $d/x = x\backslash d$ (d is cyclic) and
   $d/(x\backslash d) = (d/x)\backslash d = x$ (d is dualizing).

REMARK 4.1. If $\langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ is a dualizing *RL* and we define $\sim x = d/x$, then $\langle L, \wedge, \vee, \cdot, e, \sim\rangle$ is an involutive RL. On the other hand, if $\langle L, \wedge, \vee, \cdot, e, \sim\rangle$ is an iRL, then by defining

1. $d = \sim e$ and
2. $x\backslash z = \sim((\sim z) \cdot x)$ $z/x = \sim(x \cdot (\sim z))$

we have that $\langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ is a dualizing RL.

Let us write $\mathbf{1}$ and $\mathbf{0}$ for the greatest and least elements of a RL, L, if they exist. Then we have in L that $x \cdot \mathbf{0} = \mathbf{0} = \mathbf{0} \cdot x$; in particular, $\mathbf{1} \cdot \mathbf{0} = \mathbf{0}$. We also have that $x\backslash\mathbf{1} = \mathbf{1}$ and $\mathbf{0}\backslash x = \mathbf{1}$. Hence, $x\backslash\mathbf{1} = \mathbf{1} = \mathbf{1}/x$ and $\mathbf{0}\backslash x = \mathbf{1} = x/\mathbf{0}$.

We shall now list some immediate useful properties that holds in any involutive distributive residuated lattice:

Lemma 4.2. *For any $L \in \mathrm{iRL}$ and any $x, y \in L$, we have that,*

a. $e \leq x/x$ *and* $e \leq x \setminus x$

b. $x/y = (\sim x) \setminus (\sim y)$ *and* $x \setminus y = (\sim x)/(\sim y)$

c. $e \leq y/x$ *if and only if* $x \leq y$ *if and only if* $e \leq x \setminus y$

d. $\sim (x \vee y) = (\sim x) \wedge (\sim y)$

Proof. Properties a., b. and c. follow easily from the properties of the involution and the right and left residue. Item *d.* follows from the fact that $(\_) \to (\sim e)$ has an adjoint. ■

As an immediate consequence of d. in previous Lemma and the fact that $\sim$ is an involution, we have the following. (Recall that we are assuming that all our lattices are distributive.)

Lemma 4.3. *Let $L \in \mathrm{iRL}$. Then, $(L, \wedge, \vee, \sim, \mathbf{0}, \mathbf{1})$ is a De Morgan Algebra.*

From now on let $\mathrm{iRL_b}$ be the category of involutive residuated lattices whose underlying lattice is bounded distributive. Lemma 4.3 implies that the underlying lattice is actually a De Morgan algebra, so we have a forgetful functor $\mathrm{iRL_b} \to \mathrm{DM}$. Moreover, the (cyclic) involution of $\widehat{L}$ is given by $\sim (x_0, x_1) = (x_1, x_0)$.

Theorem 4.4 (See [9]). *Let $(L, \cdot, e, \backslash, /)$ be a residuated lattice with greatest element 1. Then the De Morgan algebra $\widehat{L}$ can be equipped with a residuated structure as follows:*

the identity element is given by $(e, 1)$

$$(x_0, x_1) * (y_0, y_1) = (x_0 \cdot y_0,\ y_1/x_0 \wedge y_0\backslash x_1)$$
$$(x_0, x_1)\backslash(y_0, y_1) = (x_0\backslash y_0 \wedge x_1/y_1,\ y_1 \cdot x_0)$$
$$(x_0, x_1)/(y_0, y_1) = (x_0/y_0 \wedge x_1\backslash y_1,\ y_0 \cdot x_1)$$

Furthermore, we have the following facts:

i. $\mathbf{d} = (1, e)$ *is a cyclic dualizing element of $\widehat{L}$. Recall that this means that* $(x_0, x_1)\backslash(1, e) = (x_1, x_0) = (1, e)/(x_0, x_1)$.

ii. $\tilde{L} = \langle L \times L^{op}, \wedge, \vee, \cdot, \backslash, /, \mathbf{e}, \mathbf{d}\rangle$ *with* $\mathbf{e} = (e, 1)$ *and* $\mathbf{d} = (1, e)$ *is a dualizing residuated lattice.*

iii. If we consider the structure $L^* = \langle L \times \{1\}, \wedge, \vee, \cdot, \backslash^*, /^*, \mathbf{e}\rangle$, *where:*
$$B/^*A = B/A \wedge (1, 1)$$
$$A\backslash^*B = A\backslash B \wedge (1, 1)$$
then, the map $\varepsilon : L \longrightarrow \widehat{L^*}$, defined by $\varepsilon(a) = (a, 1)$, for all $a \in L$ is a residuated lattice isomorphism.

LEMMA 4.5. *The functor* $\mathrm{IRL}_0 \to \mathrm{DL}$ *followed by* $\widehat{(\_)} : \mathrm{DL} \to \mathrm{DM}$ *factors through the forgetful* $\mathrm{iRL_b} \to \mathrm{DM}$.

PROOF. Let $f : L \to L'$ be in IRL_0. We need to check that $\widehat{f} = f \times f^{op}$ is a morphism of residuated lattices. But this is a straightforward computation. ∎

So we have obtained a commutative diagram

$$\begin{tikzcd} \mathrm{IRL}_0 \arrow[r, "\widehat{(\_)}"] \arrow[d] & \mathrm{iRL_b} \arrow[d] \\ \mathrm{DL} \arrow[r, "\widehat{(\_)}"'] & \mathrm{DM} \end{tikzcd}$$

with the properties discussed at the end of Section 3.

5. Centered residuated lattices

It is well known (see, for example [7]) that for any residuated lattice L, the *negative cone* of L, $L^- := \{x \in M \ : \ x \leq e\}$, is again a residuated lattice, with

$$a/^- b = (a/b) \wedge e \qquad \text{and} \qquad a \setminus^- b = (a \setminus b) \wedge e$$

and it induces a functor from the category of residuated lattices to that of integral residuated lattices. Let us denote by $\mathsf{C}_\mathbf{e}$ the functor from RL to IRL induced by $L \mapsto L^-$. By item (*iii.*) of Theorem 4.4, we see that there is an isomorphism of residuated lattices between the negative cone of $\widehat{L}$ and L itself.

"Cones" of residuated lattices will play an important role in the present paper. Given a fixed element c of a residuated lattice L, it seems difficult to have a natural picture of how the original residuated structure restricts to the cones $\{x \in L \mid x \leq c\}$ or $\{x \in L \mid x \geq c\}$. So we will usually consider cones relative to fixed elements with different properties that will allow us to obtain a nice residuated structure in the associated cone.

Let ciRL be the category of involutive residuated lattices L together with a distinguished element $\mathbf{c}$ such that $\sim \mathbf{c} = \mathbf{c}$. Morphisms are morphisms of involutive residuated lattices that preserve the distinguished element. We will usually refer to the objects of this category as *centered residuated lattices*. Let us write $\mathbf{c}\mathrm{iRL_b}$ for the subcategory of ciRL whose objects are bounded lattices.

5.1. Residuated lattices with centered unit

We can immeditatly consider the full subcategory of ciRL determined by those residuated lattices such that the distinguished center happens to be the identity $\mathbf{e}$ of the underlying residuated lattice. Let us denote the resulting subcategory by $\mathbf{e}$iRL. We also call $\mathsf{C}_{\mathbf{e}}$ the composition : $\mathbf{e}$iRL $\to$ RL $\to$ IRL; in particular, this functor restricts to one between $\mathbf{e}\text{iRL}_{\text{b}}$ and IRL_0.

5.2. Proto-integral residuated lattices

Now let us consider upper cones. We again want a functor towards IRL_0 so let us consider the full subcategory of $\mathbf{c}\text{iRL}_{\text{b}}$ determined by those objects L with a distinguished center $\mathbf{c}$ that satisfy the identities

$$1 * (\mathbf{c} \vee y) = \mathbf{c} \vee y = (\mathbf{c} \vee y) * 1$$

for every y in L. Call such objects *proto-integral*. Denote the resulting category by PRL. (It is clear that in any proto-integral L, $1 * \mathbf{c} = \mathbf{c} = \mathbf{c} * 1$.)

For $M \in$ PRL we write $\mathsf{C}M := \{x \in M \mid x \geq \mathbf{c}\}$. If we introduce the notation $x_{\mathbf{c}} = x \vee \mathbf{c}$, we can also write $\mathsf{C}M = \{x_{\mathbf{c}} \mid x \in M\}$. From the next result it follows that the assignment $M \mapsto \mathsf{C}M$ extends to a functor PRL $\to \text{IRL}_0$

Lemma 5.1. *Let M be a proto-integral residuated lattice. If we endow $\mathsf{C}M$ with the product given by*

$$x_{\mathbf{c}} *_{\mathbf{c}} y_{\mathbf{c}} = (x * y)_{\mathbf{c}} \tag{1}$$

and the operations $/, \backslash, \wedge$ and $\wedge$ inherited from M, then $(\mathsf{C}M, \wedge, \vee, *_{\mathbf{c}}, /, \backslash, \mathbf{1}, \mathbf{c})$ *is a bounded integral residuated lattice with unit* $\mathbf{1}$ *and bottom* $\mathbf{c}$*.*

Proof. It is clear that $\mathsf{C}M$ is a sublattice of M, with least element $\mathbf{c}$ and that this set is closed by $*_{\mathbf{c}}$ and the residuals of $*$. Let us now check $*_{\mathbf{c}}$ is an associative operation on $\mathsf{C}M$. Let x, y and z be elements of $\mathsf{C}M$; that is to say, $x, y, z \geq \mathbf{c}$. Then,

$$\begin{aligned}(x *_{\mathbf{c}} y) *_{\mathbf{c}} z = ((x * y) \vee \mathbf{c}) *_{\mathbf{c}} z &= [((x * y) \vee \mathbf{c}) * z] \vee \mathbf{c} = \\ &= ((x * y) * z) \vee (\mathbf{c} * z)) \vee \mathbf{c} = \\ &= ((x * y) * z) \vee \mathbf{c}\end{aligned}$$

since $\mathbf{c} * z \leq \mathbf{c} * \mathbf{1} = \mathbf{c}$. On the other hand,

$$\begin{aligned}x *_{\mathbf{c}} (y *_{\mathbf{c}} z) = (x *_{\mathbf{c}} ((y * z) \vee \mathbf{c}) &= [(x * ((y * z) \vee \mathbf{c})] \vee \mathbf{c} \\ &= [(x * (y * z)) \vee (x * \mathbf{c})] \vee \mathbf{c} = \\ &= (x * (y * z)) \vee \mathbf{c}\end{aligned}$$

Since $*$ is an associative operation, it follows the associativity of $*_{\mathbf{c}}$.

The fact that 1 is the unit of $*_{\mathbf{c}}$ is immediate from the axioms defining proto-integral lattices.

The monotony of $*_{\mathbf{c}}$ is straightforward. Let us finally see that the residues of $*$ are those of $*_{\mathbf{c}}$. On one hand, we have that

$$(x *_{\mathbf{c}} y) \leq z \Rightarrow ((x * y) \vee \mathbf{c}) \leq z \Rightarrow (x * y) \leq z$$

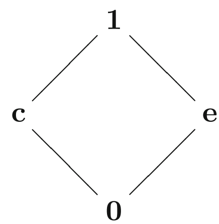
On the other, since $z = z \vee \mathbf{c}$, we have that

$$(x * y) \leq z \Rightarrow (x * y) \vee \mathbf{c} \leq z \vee \mathbf{c} = z \Rightarrow (x *_{\mathbf{c}} y) \leq z \qquad \blacksquare$$

5.3. Complemented-unit involutive rl's

In this section we start the work towards finding the largest convenient subcategory of $\mathrm{iRL_b}$ with respect to $\widehat{(\_)} : \mathrm{IRL_0} \to \mathrm{iRL_b}$. Again following the idea suggested in Section 2 we look at the initial object $\mathbf{2}$ in $\mathrm{IRL_0}$. That is, the chain with two elements seen as a residuated lattice with $\cdot = \wedge$.

EXAMPLE 5.2. The involutive $\widehat{\mathbf{2}}$ is the residuated lattice:



with $\mathbf{e} = (1, 1)$, $\mathbf{c} = (0, 0)$ and the product and involution completely determined by the conditions that $\mathbf{e}$ is the unit of the monoid, $\sim \mathbf{e} = \mathbf{e}$, $\mathbf{c}^2 = \mathbf{c} * \mathbf{c} = \mathbf{0}$ and the fact of having a De Morgan algebra reduct.

Now, what are the objects X in $\mathrm{iRL_b}$ for which there exists a unique morphism $\widehat{\mathbf{2}} \to X$? This question leads us to define the following.

DEFINITION 5.3. A *complemented-unit involutive residuated lattice* is an algebra $(L, \wedge, \vee, *, /, \backslash, \sim, \mathbf{e}, \mathbf{c}, \mathbf{0}, \mathbf{1})$, such that,

1. $(L, \wedge, \vee, *, /, \backslash, \sim, \mathbf{e})$ is a (cyclic) involutive residuated lattice,
2. L is bounded as a distributive lattice,
3. $\sim \mathbf{e} = \mathbf{e}$ and
4. $\mathbf{e}$ has a complement $\mathbf{c}$ in L

We shall write cuRL for the full subcategory of $\mathrm{iRL_b}$ whose objects are complemented-unit iRL$'s$. This category is mainly a useful intermediate step where some axioms and their relations can be clearly stated.

LEMMA 5.4. *In any* cuRL*, we have that,*

a. $\sim \mathbf{c} = \mathbf{c}$,

b. $\mathbf{c} * \mathbf{c} \leq \mathbf{e}$ *and*

c. There are equivalent,

   i. $\mathbf{c} * \mathbf{c} = 0$ *and*

   ii. $\mathbf{c} * \mathbf{1} = \mathbf{c} = \mathbf{1} * \mathbf{c}$.

PROOF. We have that in any $L \in$ cuRL, $\sim \mathbf{e} = \mathbf{e}$. Then $\mathbf{0} =\sim (\mathbf{e} \vee \mathbf{c}) =\sim \mathbf{e} \wedge \sim \mathbf{c} = \mathbf{e} \wedge \sim \mathbf{c}$ and $\mathbf{1} =\sim (\mathbf{e} \wedge \mathbf{c}) =\sim \mathbf{e} \vee \sim \mathbf{c} = \mathbf{e} \vee \sim \mathbf{c}$. Hence $\sim \mathbf{c}$ is a complement of $\mathbf{e}$, and since L is distributive, $\sim \mathbf{c} = \mathbf{c}$.

Since in any $L \in$ cuRL it holds that $\sim \mathbf{c} = \mathbf{e}/\mathbf{c}$, it follows that $\mathbf{c} * (\sim \mathbf{c}) \leq \mathbf{e}$. Then, by (a.), we get that $\mathbf{c}^2 \leq \mathbf{e}$.

On one hand, suppose that $\mathbf{c}^2 = \mathbf{0}$. Then $\mathbf{c}*\mathbf{1} = \mathbf{c}*(\mathbf{c}\vee\mathbf{e}) = \mathbf{c}^2\vee(\mathbf{c}*\mathbf{e}) = \mathbf{c}$; and similarly for the left side multiplication by $\mathbf{1}$. On the other hand, if $\mathbf{c} * \mathbf{1} = \mathbf{c}$, then $\mathbf{c}^2 \vee \mathbf{c} = \mathbf{c}$, and hence, $\mathbf{c}^2 \leq \mathbf{c}$. Since $\mathbf{c}^2 \leq \mathbf{e}$, we have that $\mathbf{c}^2 \leq \mathbf{c} \wedge \mathbf{e} = \mathbf{0}$. ■

5.4. The axiom $\mathbf{c} * \mathbf{c} = \mathbf{0}$

Clearly, the objects X in $\mathrm{iRL_b}$ for which there exists a unique $\widehat{\mathbf{2}} \to X$ are exactly the complemented-unit involutive irls such that $\mathbf{c} * \mathbf{c} = \mathbf{0}$. But it is fair to ask if this condition follows from the axioms defining complemented-unit involutive irls. The answer is negative as the following example shows.

EXAMPLE 5.5. Consider the De Morgan lattice of Example 5.2, but with the product given in the table,

$*$	$\mathbf{0}$	$\mathbf{c}$	$\mathbf{e}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{c}$	$\mathbf{0}$	$\mathbf{e}$	$\mathbf{c}$	$\mathbf{1}$
$\mathbf{e}$	$\mathbf{0}$	$\mathbf{c}$	$\mathbf{e}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

It can be seen that this is a complemented-unit iRL where $\mathbf{c} * \mathbf{c} \neq \mathbf{0}$.

Let NRL be the full subcategory of cuRL whose objects satisfy the equation $\mathbf{c} * \mathbf{c} = \mathbf{0}$. The objects of this category may be called *nil-complemented-unit* residuated lattices. Pictorially, we have introduced the following subcategories of centered residuated lattices of $\mathrm{iRL_b}$:

$$\begin{tikzcd}
\mathrm{NRL} \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\equiv"] & \mathrm{PRL} \arrow[dr] & & \\
\mathrm{cuRL} \arrow[r] & \mathbf{e}\mathrm{iRL_b} \arrow[r] & \mathbf{c}\mathrm{iRL_b} \arrow[r] & \mathrm{iRL_b}
\end{tikzcd}$$

where the the functors $\mathrm{cuRL} \to \mathbf{e}\mathrm{iRL_b}$ and $\mathrm{NRL} \to \mathrm{PRL}$ are the obvious functors distinguishing the identity and its complement respectively. Post-composing the two full subcategories $\mathrm{NRL} \to \mathbf{c}\mathrm{iRL_b}$ with the forgetful functor produces a full subcategory $\mathrm{NRL} \to \mathrm{iRL_b}$.

LEMMA 5.6. *The functor* $\widehat{(\_)} : \mathrm{IRL_0} \to \mathrm{iRL_b}$ *factors through the embedding* $\mathrm{NRL} \to \mathrm{iRL_b}$.

PROOF. Much as in Example 5.2, one has for every L in $\mathrm{IRL_0}$ that, in $\widehat{L}$, $\mathbf{e} = (1,1)$ has a complement $\mathbf{c} = (0,0)$. ■

We are now facing a commutative diagram

$$\begin{tikzcd}
\mathrm{IRL_0} \arrow[d] \arrow[dr, "\widehat{(_)}"] & \\
\mathrm{NRL} \arrow[r] & \mathrm{iRL_b}
\end{tikzcd}$$

and in Section 6 we show that $\mathrm{NRL} \to \mathrm{iRL_b}$ is the largest convenient subcategory (with respect to $\widehat{(\_)} : \mathrm{IRL_0} \to \mathrm{iRL_b}$) and moreover, that the adjunction $\widehat{(\_)} : \mathrm{IRL_0} \to \mathrm{NRL}$ is actually an equivalence. So, much as in the case of the original Kalman situation, the right adjoint associated to the largest convenient subcategory has a further right adjoint. In the present case, though, the largest convenient subcategory is, in the precise sense stated above, also the smallest one.

5.5. The cones of an object in NRL

In this section we show that for an object L in NRL, the cones $\mathsf{C}L$ and $\mathsf{C}_{\mathbf{e}}L$ are essentially the same. First notice that NRL appears as an evident full subcategory of $\mathbf{e}\mathrm{iRL_b}$. Slightly less evidently, it appears as a full subcategory of PRL.

LEMMA 5.7. *Let $M \in$ NRL and $\mathsf{C}M$ as above. If we distinguish the complement $\mathbf{c}$ of the identity $\mathbf{e}$ then we obtain a proto-integral residuated lattice.*

PROOF. Now let $x \geq \mathbf{c}$. We can calculate as follows

$$x * \mathbf{1} = x * (\mathbf{c} \vee \mathbf{e}) = (x * \mathbf{c}) \vee (x * \mathbf{e}) = (x * \mathbf{c}) \vee x =$$
$$= (x * \mathbf{c}) \vee (\mathbf{c} \vee x) = ((x * \mathbf{c}) \vee \mathbf{c}) \vee x = \mathbf{c} \vee x = x$$

using that $\mathbf{c} * 1 = \mathbf{c}$ in order to have $(x * \mathbf{c}) \vee \mathbf{c} = \mathbf{c}$. Hence, $x *_c \mathbf{1} = (x * \mathbf{1}) \vee \mathbf{c} = x \vee \mathbf{c} = x$. Similarly we can check that $1 *_{\mathbf{c}} x = x$. So the result follows. ■

REMARK 5.8. Observe that when $\mathbf{c}^2 = \mathbf{0}$ holds in the algebra, we get,

CE. For any $x, y \in L$, $(x * y) \wedge \mathbf{e} = (x \wedge \mathbf{e}) * (y \wedge \mathbf{e})$.

CC. For any $x, y \in L$, $(x * y) \wedge \mathbf{c} = ((x \wedge \mathbf{c}) * y) \vee (x * (y \wedge \mathbf{c}))$.

LEMMA 5.9. *Let $M \in$ NRL. Then there is an isomorphism of residuated lattices $\alpha : \mathsf{C}M \to \mathsf{C}_{\mathbf{e}}M$.*

PROOF. Let $M \in$ NRL. For any $x_{\mathbf{c}} \in M$, define $\alpha(x_{\mathbf{c}}) := x_{\mathbf{c}} \wedge \mathbf{e}$. It is clear that α is a map from $\mathsf{C}M$ to $\mathsf{C}_{\mathbf{e}}M$. Since M is a distributive lattice, it is also immediate that α is an isomorphism of lattices between $\mathsf{C}M$ and $\mathsf{C}_{\mathbf{e}}M$. Let us see that it is also a residuated lattices morphism.

Let $x, y \in M$. We have that

$$\alpha(x_{\mathbf{c}} *_{\mathbf{c}} y_{\mathbf{c}}) = [((x \vee \mathbf{c}) * (y \vee \mathbf{c})) \vee c] \wedge e = ((x \vee \mathbf{c}) * (y \vee \mathbf{c})) \wedge e =$$
$$= ((x \vee \mathbf{c}) \wedge e) * ((y \vee \mathbf{c}) \wedge e) = \alpha(x_{\mathbf{c}}) * \alpha(y_{\mathbf{c}})$$

Then, α preserves the product.

On the other hand, We have that

$$\alpha(y_{\mathbf{c}}/x_{\mathbf{c}}) = [\sim ((x \vee \mathbf{c}) * \sim (y \vee \mathbf{c}))] \wedge e =$$
$$= [\sim ((x \vee \mathbf{c}) * (\sim y \wedge \mathbf{c}))] \wedge e = [\sim ((x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{c}))] \wedge e$$

and that

$$\alpha(y_{\mathbf{c}})/^{\mathbf{e}}\alpha(x_{\mathbf{c}}) = (y \wedge \mathbf{e})/^{\mathbf{e}}(x \wedge \mathbf{e}) = [(y \wedge \mathbf{e})/(x \wedge \mathbf{e})] \wedge \mathbf{e} =$$
$$= [\sim ((x \wedge \mathbf{e}) * \sim (y \wedge \mathbf{e}))] \wedge \mathbf{e} =$$
$$= [\sim ((x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{c}) \vee (x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{e}) \vee (x \wedge \mathbf{e}))] \wedge \mathbf{e} =$$
$$= \sim ((x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{c})) \wedge \sim ((x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{e}) \vee (x \wedge \mathbf{e})) \wedge \mathbf{e} =$$
$$= \sim ((x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{c})) \wedge \mathbf{e}$$

where we have used that $(x \vee \mathbf{c}) = [(x \wedge \mathbf{c}) \vee (x \wedge \mathbf{e})] \vee \mathbf{c} = \mathbf{c} \vee (x \wedge \mathbf{e})$ and that $(x \wedge \mathbf{e}) * (\sim y \wedge \mathbf{e}) \vee (x \wedge \mathbf{e}) \leq \mathbf{e}$ respectively. Hence $\alpha(y_{\mathbf{c}}/x_{\mathbf{c}}) = \alpha(y_{\mathbf{c}})/^{\mathbf{e}}\alpha(x_{\mathbf{c}})$.

Similarly, we obtain that $\alpha(y_{\mathbf{c}} \setminus x_{\mathbf{c}}) = \alpha(y_{\mathbf{c}}) \setminus^{\mathbf{e}} \alpha(x_{\mathbf{c}})$. ■

6. The equivalence $\widehat{(\_)} : \mathrm{IRL}_0 \to \mathrm{NRL}$

The main point of this section is to show that functor $\widehat{(\_)} : \mathrm{IRL}_0 \to \mathrm{NRL}$ is actually an equivalence.

Recall that we are assuming that all lattices that appear are distributive. When we write x for an element of $L \times L^{op}$ we shall mean that $x = (x_0, x_1)$ without explicitly mention it.

LEMMA 6.1. *Let* $L \in \mathrm{cuRL}$. *It holds that,*

CE'. For any $x, y \in L$, $(x * y) \wedge \mathbf{e} = ((x \wedge \mathbf{e}) * (y \wedge \mathbf{e})) \vee ((x \wedge \mathbf{c}) * (y \wedge \mathbf{c}))$.

CC'. For any $x, y \in L$, $(x * y) \wedge \mathbf{c} = ((x \wedge \mathbf{c}) * (y \wedge \mathbf{e})) \vee ((x \wedge \mathbf{e}) * (y \wedge \mathbf{c}))$.

PROOF. Let us first observe that for any $x \in L$, we can write $x = x \wedge \mathbf{1} = x \wedge (\mathbf{e} \vee \mathbf{c}) = (x \wedge \mathbf{e}) \vee (x \wedge \mathbf{c})$. For brevity we shall write $x_0 := x \wedge \mathbf{c}$ and $x_1 := x \wedge \mathbf{e}$.

Let us now take $x, y \in L$. We have that

$$x * y = (x_0 \vee x_1) * (y_0 \vee y_1) = (x_0 * y_0) \vee (x_0 * y_1) \vee (x_1 * y_0) \vee (x_1 * y_1)$$

Since $x_0 * y_0 \leq \mathbf{c} * \mathbf{c} \leq \mathbf{e}$, $x_0 * y_1 \vee x_1 * y_0 \leq (\mathbf{c} * \mathbf{e}) \vee (\mathbf{e} * \mathbf{c}) = \mathbf{c}$ and $x_1 * y_1 \leq \mathbf{e} * \mathbf{e} = \mathbf{e}$, we conclude that

$$(x * y)_0 = (x_0 * y_1) \vee (x_1 * y_0) \tag{2}$$

and

$$(x * y)_1 = (x_0 * y_0) \vee (x_1 * y_1) \tag{3}$$

Equation (2) is exactly that in (CC') and equation (3) that in (CE'). ■

REMARK 6.2. Condition (CE) of Remark 5.8 is an immediate consequence of (CE'). On the other hand, let us note that $x * y_0 = (x_0 \vee x_1) * y_0 = x_0 * y_0 \vee x_1 * y_0 = \mathbf{0} \vee x_1 * y_0 = x_1 * y_0$ and $x_0 * y = x_0 * y_1$. Thus, after replacing in (2) we get

$$(x * y)_0 = x_0 * y \vee x * y_0$$

which is the equation in (CC) of Remark 5.8.

THEOREM 6.3. *Functor* $\widehat{(\_)} : \mathrm{IRL}_0 \to \mathrm{NRL}$ *is left adjoint to* $\mathsf{C}_{\mathbf{e}} : \mathrm{NRL} \to \mathrm{IRL}_0$. *Furthermore, they are an equivalence pair.*

Proof. For each $M \in$ NRL, let us take the map $\varepsilon_M : \widehat{\mathsf{C_e}M} \to M$ given by

$$\varepsilon_M(x, y) := x \vee ((\sim y) \wedge \mathbf{c}) \tag{4}$$

for any $x, y \in M$.

Let us now see that it is in fact a map in NRL. Let $(a, x), (b, y) \in \widehat{\mathsf{C_e}M}$, then we have,

$$\begin{aligned}\varepsilon_M((a, x) \wedge (b, y)) &= \varepsilon_M((a \wedge b, x \vee y)) = \\ &= (a \wedge b) \vee [\sim (x \vee y) \wedge \mathbf{c}] = \\ &= (a \wedge b) \vee [(\sim x) \wedge (\sim y) \wedge \mathbf{c}]\end{aligned}$$

and that,

$$\begin{aligned}&\varepsilon_M(a, x) \wedge \varepsilon_M(b, y) = [a \vee (\sim x \wedge \mathbf{c})] \wedge [b \vee (\sim y \wedge \mathbf{c})] = \\ &= (a \wedge b) \vee ((\sim x \wedge \mathbf{c}) \wedge b) \vee (a \wedge (\sim y \wedge \mathbf{c})) \vee [(\sim x \wedge \mathbf{c}) \wedge (\sim y \wedge \mathbf{c})] = \\ &= (a \wedge b) \vee [(\sim x) \wedge (\sim y) \wedge \mathbf{c}]\end{aligned}$$

since $(\sim x \wedge \mathbf{c}) \wedge b = a \wedge (\sim y \wedge \mathbf{c}) = \mathbf{0}$ because $a, b \leq \mathbf{e}$ and $\mathbf{c} \wedge \mathbf{e} = \mathbf{0}$. Thus,

$$\varepsilon_M((a, x) \wedge (b, y)) = \varepsilon_M(a, x) \wedge \varepsilon_M(b, y)$$

Similar computations show that

$$\begin{aligned}&\varepsilon_M((a, x) \vee (b, y)) = \varepsilon_M(a, x) \vee \varepsilon_M(b, y) \\ &\varepsilon_M(\sim (a, x)) = \sim \varepsilon_M(a, x) \\ &\varepsilon_M(\mathbf{e}, \mathbf{e}) = \mathbf{e} \text{ and} \\ &\varepsilon_M(\mathbf{0}, \mathbf{0}) = \mathbf{c}\end{aligned}$$

Let us finally check that

$$\varepsilon_M((a, x) * (b, y)) = \varepsilon_M(a, x) * \varepsilon_M(b, y) \tag{5}$$

We start by observing that, since $a, b, x, y \in M$, in this particular case we can write

$$(a, x) * (b, y) = (a * b, \ \sim [(a * (\sim y)) \vee ((\sim x) * b)])$$

Hence, we have on one hand that,

$$\begin{aligned}\varepsilon_M((a, x) * (b, y)) &= (a * b) \vee [((a * (\sim y)) \vee ((\sim x) * b)) \wedge \mathbf{c}] = \\ &= (a * b) \vee ((a * (\sim y)) \wedge \mathbf{c}) \vee (((\sim x) * b)) \wedge \mathbf{c}) = \\ &= (a * b) \vee (a * ((\sim y) \wedge \mathbf{c})) \vee (((\sim x) \wedge \mathbf{c}) * b)\end{aligned}$$

Last equality follows from condition (CC) of Remark 5.8 and the fact that $a, b \leq \mathbf{e}$.

On the other hand, we have that:

$$\varepsilon_M(a,x) * \varepsilon_M(b,y) = (a \vee ((\sim x) \wedge \mathbf{c})) * (b \vee ((\sim y) \wedge \mathbf{c})) =$$
$$(a * b) \vee [a * ((\sim y) \wedge \mathbf{c})] \vee [((\sim x) \wedge \mathbf{c}) * b] \vee [((\sim y) \wedge \mathbf{c})) * ((\sim x) \wedge \mathbf{c})] =$$
$$(a * b) \vee (a * ((\sim y) \wedge \mathbf{c})) \vee (((\sim x) \wedge \mathbf{c}) * b)$$

Thus, (5) holds.

Let us now define for any $L \in \mathrm{IRL}_0$, $\eta_L : L \to \mathsf{C}_{\mathbf{e}}(\widehat{L})$ by

$$\eta_L(a) := (a, 1) \tag{6}$$

By Theorem 4.4 *(iii)*, η_L is an isomorphism in IRL_0.

We claim that η and ε are the unit and counit of an adjunction $\widehat{(\_)} \dashv \mathsf{C}_{\mathbf{e}}$.

Straightforward computations show the naturality of both η and ε. We shall now check that for any $L \in \mathrm{IRL}_0$ and any $M \in \mathrm{NRL}$, $\varepsilon_{\widehat{L}}(\widehat{\eta_L}) = 1_{\widehat{L}}$ and $\mathsf{C}_{\mathbf{e}}(\varepsilon_M)\eta_{\mathsf{C}_{\mathbf{e}}M} = 1_{\mathsf{C}_{\mathbf{e}}M}$.

Let $(a,b) \in \widehat{L}$, $\widehat{\eta_L}(a,b) = ((a,1),(b,1))$, and

$$\varepsilon_{\widehat{L}}((a,1),(b,1)) = (a,1) \vee [\sim (b,1) \wedge (0,0)] = (a,1) \vee (0,b) = (a,b)$$

Let $x \wedge \mathbf{e} \in \mathsf{C}_{\mathbf{e}}M$, $\eta_{\mathsf{C}_{\mathbf{e}}M}(x \wedge \mathbf{e}) = (x \wedge \mathbf{e}, \mathbf{e})$, and

$$\mathsf{C}_{\mathbf{e}}(\varepsilon_M)(x \wedge \mathbf{e}, \mathbf{e}) = (x \wedge \mathbf{e}) \vee (\mathbf{e} \wedge \mathbf{c}) = x \wedge \mathbf{e}$$

Thus, we have proved that $\widehat{(\_)}$ is left adjoint to $\mathsf{C}_{\mathbf{e}}$. Furthermore, since both η_L and ε_M are isos, they are an equivalence. ■

Corollary 6.4. *Functor* $\mathsf{C} : \mathrm{NRL} \to \mathrm{IRL}_0$ *is an adjoint equivalence.*

Proof. Immediate from Lemma 5.9 and Theorem 6.3. ■

7. Residuated Kalman construction

In Section 3 we defined $\mathsf{K} : \mathrm{DL} \to \mathrm{DM}$ as assigning to L, the centered Kleene algebra $\mathsf{K}L$ with underlying set $\{(x,y) \in \widehat{L} \mid x \wedge y = 0\}$ and lattice operations inherited from $\widehat{L}$.

Let us now proceed analogously but replacing $\widehat{(\_)} : \mathrm{DL} \to \mathrm{DM}$ by its lifting $\widehat{(\_)} : \mathrm{IRL}_0 \to \mathrm{iRL}_{\mathrm{b}}$ and $\wedge$ by a residuated operation. More precisely, for L in IRL_0 we define $\mathsf{K}^{\bullet}L$ to be the set $\{(x,y) \in L \times L \mid x \cdot y = 0\}$.

When equipped with the partial order inherited from $\widehat{L}$, we obtain a sub-De-Morgan algebra $\mathsf{K}^{\bullet}L \to \widehat{L}$. Moreover, it is closed under the monoid operation of $\widehat{L}$ in $\mathrm{iRL_b}$ but notice that the unit $(1,1) \in \widehat{L}$ is not an element of $\mathsf{K}^{\bullet}L$. It is not difficult to check, though, that the upper bound $(1,0)$ happens to be neutral with respect to the restricted monoid operation. In this way, $\mathsf{K}^{\bullet}L$ can be equipped with an integral and involutive residuated lattice structure. From now on, $\mathsf{K}^{\bullet}L$ will always be considered equipped with this structure. Further standard consideration of morphisms allows us to deduce the following result.

LEMMA 7.1. *The assignment* $L \mapsto \mathsf{K}^{\bullet}L$ *extends to a functor* $\mathsf{K}^{\bullet} : \mathrm{IRL_0} \to \mathrm{iRL_b}$.

We see the functor $\mathsf{K}^{\bullet} : \mathrm{IRL_0} \to \mathrm{iRL_b}$ as the first step towards an analogue of Kalman's construction for residuated lattices. We feel, though, that $\mathsf{K}^{\bullet} : \mathrm{IRL_0} \to \mathrm{iRL_b}$ is not the right place to stop. Much as Kalman saw his construction as producing a Kleene algebra and not *just* a De Morgan algebra, we want a more subtle description of $\mathsf{K}^{\bullet}L$. The first thing to notice is that $\mathsf{K}^{\bullet}L$ is integral.

Deeper insight is provided by looking at $\mathsf{K}^{\bullet}L$ from the perspective of what we have learned about $\widehat{(\_)} : \mathrm{IRL_0} \to \mathrm{iRL_b}$ and Theorem 6.3. The latter result characterized the equations that hold in all objects of the form $\widehat{L}$ as those that follow from the equations we used to present the algebraic category NRL. These equations postulated the existence of a complement $\mathbf{c}$ of the unit $\mathbf{e}$ and the requirement that $\mathbf{e}$ (an so, also $\mathbf{c}$) be a center. Now, the element $\mathbf{e} = (1,1)$ will not be in $\mathsf{K}^{\bullet}L$ unless L is trivial, but $\mathbf{c} = (0,0)$ is always in $\mathsf{K}^{\bullet}L$. Moreover, it is clear from the proof of Theorem 6.3 that (CE') and (CC') are essential properties of $\widehat{L}$. This suggests that we consider condition (CC) (see Remark 5.8).

DEFINITION 7.2. Let $(L, \vee, \wedge)$ be a lattice equipped with a binary operation $* : L \times L \to L$. We call $c \in L$ a *Leibniz element* if for any $x, y \in L$, $(x * y) \wedge c = ((x \wedge c) * y) \vee (x * (y \wedge c))$.

The terminology is chosen to reflect the analogy with the second item in Definition 7.2 and the usual Leibniz rule for differentiation of a product.

DEFINITION 7.3. A $\mathbf{c}$-*differential lattice* is an integral centered residuated lattice whose distinguished center $\mathbf{c}$ is a Leibniz element. A *morphism of* $\mathbf{c}$-*differential lattices* is a morphism of residuated lattices which preserves the distinguished Leibniz element.

Let us denote de category of $\mathbf{c}$-differential lattices by DRL. It is clearly an algebraic category and there are obvious forgetful functors $\mathrm{DRL} \to \mathrm{iRL_b}$ and $\mathrm{DRL} \to \mathrm{IRL_0}$. Moreover, by taking $\mathbf{c} = (0, 0)$, we obtain a factorization of $\mathsf{K}^\bullet$ through DRL as below

$$\begin{tikzcd} \mathrm{IRL_0} \arrow[r] \arrow[dr, "\mathsf{K}^\bullet"'] & \mathrm{DRL} \arrow[d] \\ & \mathrm{iRL_b} \end{tikzcd}$$

and we denote the factorization by $\mathsf{K}^\bullet : \mathrm{IRL_0} \to \mathrm{DRL}$ again.

It seems relevant to notice that while in Section 6 the condition $\mathbf{c} * \mathbf{c} = \mathbf{0}$ played an important role, Definitions 7.2 and 7.3 do not mention this equality. The reason is the following.

LEMMA 7.4. *If L is an integral involutive residuated lattice and $c \in L$ is such that $\sim c = c$ then $c \cdot c = 0$.*

PROOF. The fact that L is integral implies that $c/c = 1$. Then we can calculate: $c \cdot c = c \cdot (\sim c) =\sim (c/c) =\sim 1 = 0$. ■

For any $A \in \mathrm{DRL}$, we take $\mathsf{C}A := \{a \in A \ : \ a \geq \mathbf{c}\}$ with the product defined in Lemma 5.1. This defines another functor $\mathsf{C} : \mathrm{DRL} \to \mathrm{IRL_0}$.

First, for any c-differential lattice T, let the morphism $\eta_T : T \to \mathsf{K}^\bullet\mathsf{C}T$ be defined by $\eta_T z = (z \vee \mathbf{c}, (\sim z) \vee \mathbf{c})$.

LEMMA 7.5. *The function η_T is an injective morphism of c-differential lattices. Moreover, it induces a natural transformation $Id \to \mathsf{K}^\bullet\mathsf{C}$.*

PROOF. It is straightforward to check that η_T is a morphism of De Morgan algebras. To check that it preserves monoid structures calculate:

$$(\eta_T u) * (\eta_T v) = (u \vee \mathbf{c}, (\sim u) \vee \mathbf{c}) * (v \vee \mathbf{c}, (\sim v) \vee \mathbf{c}) =$$

$$= ((u \vee c) *_{\mathbf{c}} (v \vee \mathbf{c}), [((\sim v) \vee \mathbf{c})/^{\mathbf{c}}(u \vee \mathbf{c})] \wedge [(v \vee \mathbf{c})\backslash^{\mathbf{c}}((\sim u) \vee \mathbf{c})]) =$$

$$= ((u * v) \vee \mathbf{c}, [((\sim v) \vee \mathbf{c})/(u \vee \mathbf{c})] \wedge [(v \vee \mathbf{c})\backslash((\sim u) \vee \mathbf{c})]) =$$

$$= ((u * v) \vee \mathbf{c}, \sim [(u \vee \mathbf{c}) * (v \wedge \mathbf{c})] \wedge \sim [(u \wedge \mathbf{c}) * (v \vee \mathbf{c})]) =$$

$$= ((u * v) \vee \mathbf{c}, \sim [(u * (v \wedge \mathbf{c})) \vee ((u \wedge \mathbf{c}) * v)]) =$$

$$= ((u * v) \vee \mathbf{c}, \sim [(u * v) \wedge \mathbf{c}]) = ((u * v) \vee \mathbf{c}, \sim (u * v) \vee \mathbf{c}) = \eta_T(u * v)$$

The fact that η_T is injective morphism is deduced as in Kalman's traditional result. Finally, naturality is also straightforward. ■

THEOREM 7.6. *The functor* $\mathsf{C} : \mathrm{DRL} \to \mathrm{IRL}_0$ *is left adjoint to the functor* $\mathsf{K}^\bullet : \mathrm{IRL}_0 \to \mathrm{DRL}$. *Moreover,* $\mathsf{K}^\bullet$ *is full and faithful and* C *is faithful.*

PROOF. For any R in IRL_0, let $\varepsilon_R : \mathsf{CK}^\bullet R \to R$ be defined by $\varepsilon_R(x, 0) = x$. It is obviously an iso and it is easy to check that ε_R is a natural transformation $\mathsf{CK}^\bullet \to Id$. We show that η (discussed in Lemma 7.5) and ε are the unit and conunit of the adjunction. So let us consider the triangular identities. First we prove that for every **c**-differential lattice T, $(\mathsf{K}^\bullet\varepsilon_R)(\eta_{\mathsf{K}^\bullet R}) = id_{\mathsf{K}^\bullet R}$:

$$(\mathsf{K}^\bullet\varepsilon_R)((\eta_{\mathsf{K}^\bullet R})(x, y)) = (\mathsf{K}^\bullet\varepsilon_R)((x, y) \vee (0, 0), (y, x) \vee (0, 0)) =$$

$$= (\mathsf{K}^\bullet\varepsilon_R)((x, 0), (y, 0)) = (\varepsilon_R(x, 0), \varepsilon_R(y, 0)) = (x, y)$$

and finally, we show that, for every **c**-differential lattice T, $(\varepsilon_{\mathsf{C}T})(\mathsf{C}\eta_T) = id_{\mathsf{C}T}$. So let $x \in \mathsf{C}T$ and calculate:

$$(\varepsilon_{\mathsf{C}T})(\mathsf{C}\eta_T)x = (\varepsilon_{\mathsf{C}T})(\eta_T x) = \varepsilon_{\mathsf{C}T}(x \vee \mathbf{c}, (\sim x) \vee \mathbf{c}) = \varepsilon_{\mathsf{C}T}(x, \mathbf{c}) = x$$

So the two triangular identities hold. The fact that $\mathsf{K}^\bullet$ is full and faithful follows from the fact that the counit is iso. The fact that C is faithful follows from the fact that the unit is mono. ■

Much as in the case of Theorem 3.2, the left adjoint to $\mathsf{K}^\bullet$ has a further left adjoint. Unfortunately we do not have such a simple description as that of M. So we will have to address the problem from a different standpoint.

One way to obtain the left adjoint to $\mathsf{C} : \mathrm{DRL} \to \mathrm{IRL}_0$ is to apply Freyd's Adjoint Functor Theorem by observing that C preserves limits and that the solution set condition holds. A more explicit description can be given as follows. For any L in IRL_0, let $\mathsf{M}^\bullet L$ be the subalgebra of $\mathsf{K}^\bullet L$ generated by the elements of the form $(x, 0)$ with x in L. More intuitively, $\mathsf{M}^\bullet L$ is the smallest subalgebra of $\mathsf{K}^\bullet L$ generated by L. The assignment $L \mapsto \mathsf{M}^\bullet L$ actually extends to a functor $\mathrm{IRL}_0 \to \mathrm{DRL}$ and it turns out that it is left adjoint to $\mathsf{C} : \mathrm{DRL} \to \mathrm{IRL}_0$. Of course, one can prove this directly but it seems worth placing the result in a more general context. This context allows to see more clearly what are the features of the adjunction $\mathsf{C} \dashv \mathsf{K}^\bullet$ that allow C to have a left adjoint. The general result and its proof appears in Appendix A.

COROLLARY 7.7. *The functor* $\mathsf{C} : \mathrm{DRL} \to \mathrm{IRL}_0$ *has a left adjoint.*

PROOF. Define $\kappa : \mathsf{UC} \to \mathsf{U}$ as the inclusion of the cone. It is straightfoward to check that the hypothesis of Proposition A.4 hold. ■

Also in a general categorical context one can prove that the type of situation that arises in Section 3 and here allows to transfer congruences in the sense that the lattice of congruences of $\mathsf{C}X$ is iso to that of X and that the lattice of congruences of L is iso to that of $\mathsf{K}^{\bullet}L$. (See Appendix B for a more precise statement and proofs.)

7.1. The image of $\mathsf{K}^{\bullet}$

We now give a characterization of the image of $\mathsf{K}^{\bullet}$. Consider the following condition, which is an algebraic alternative to interpolation property as requested in [3]:

$(CK^{\cdot})$ For every pair of elements $x, y \in T$, such that $x, y \geq c$ and $x * y \leq c$ there exists $z \in T$ such that $z \vee c = x$, $\sim z \vee c = y$.

Let DRL$'$ be the full subcategory of DRL determined by those objects which satisfy $(CK^{\cdot})$.

Corollary 7.8. *The adjunction* $\mathsf{C} \dashv \mathsf{K}^{\bullet} : \mathrm{IRL}_0 \to \mathrm{DRL}$ *restricts to an equivalence* $\mathsf{C} \dashv \mathsf{K}^{\bullet} : \mathrm{IRL}_0 \to \mathrm{DRL}'$.

Proof. One shows that for every L in IRL_0, $\mathsf{K}^{\bullet}L$ satisfies $(CK^{\cdot})$ and that for T in DRL$'$, $\eta : T \to \mathsf{K}^{\bullet}\mathsf{C}T$ is surjective. Since it is also injective (Lemma 7.5), it is an iso. So both the unit and the counit of the restricted adjunction are isos. Hence the restricted adjunction is an equivalence. ■

7.2. A relation between $\widehat{(\_)}$ and $\mathsf{K}^{\bullet}$

We present an interesting factorization of $\mathsf{K}^{\bullet}$ through $\widehat{(\_)}$. This factorization stands on the following observation.

Lemma 7.9. *Let* $L \in \mathrm{IRL}_0$ *and* $x = (x_0, x_1) \in \widehat{L}$. *There are equivalent,*

a. $x * \mathbf{1} = x = \mathbf{1} * x$,

b. $\mathbf{0}/x = x = x \setminus \mathbf{0}$ *and*

c. $x_0 x_1 = x_1 x_0 = 0$

Proof. We have that $\mathbf{0}/x = \sim (x * \sim \mathbf{0}) = \sim (x * \mathbf{1})$ and $x \setminus \mathbf{0} = \sim (\sim \mathbf{0} * x) = \sim (\mathbf{1} * x)$. Hence, $\mathbf{0}/x = x$ if and only if $x * \mathbf{1} = x$ and $x \setminus \mathbf{0} = x$ if and only if $\mathbf{1} * x = x$. We have shown the equivalence between (a.) and (b.).

On the other hand, we have that

$$\begin{aligned} x * \mathbf{1} &= (x_0, x_1) * (1, 0) = (x_0, 0 \setminus x_0 \wedge x_1) && \text{and} \\ \mathbf{1} * x &= (1, 0) * (x_0, x_1) = (x_0, x_1 \wedge x_0 \setminus 0) \end{aligned}$$

Hence,

$$\begin{aligned} 0 \setminus x_0 \wedge x_1 = x_1 \quad &\text{iff} \quad x_1 \leq 0 \setminus x_0 \quad \text{iff} \quad x_1\, x_0 = 0 && \text{and} \\ x_1 \wedge x_0 \setminus 0 = x_1 \quad &\text{iff} \quad x_1 \leq x_0 \setminus 0 \quad \text{iff} \quad x_0\, x_1 = 0 \end{aligned}$$

Thus, we also have that (a.) and (c.) are equivalent. ■

The result immediately suggests considering, for a residuated lattice M with $\mathbf{1}$, the set $\mathsf{N}M := \{x \in M \ : \ x * \mathbf{1} = x = \mathbf{1} * x\}$.

Lemma 7.10. *If M is a bounded involutive residuated lattice then the subset $\mathsf{N}(M)$ of M is closed by $\wedge$, $\vee$, $\sim$ and $*$. Moreover, 1 is a neutral element and in this way $\mathsf{N}M$ becomes an integral involutive residuated lattice.*

Proof. We briefly explain how $\mathsf{N}M$ is closed under $\sim$ and leave the rest of the details for the reader. Suppose that $x * \mathbf{1} = x$ then $x \leq x/\mathbf{1}$ and hence $\sim (x/\mathbf{1}) \leq \sim x$. Since $\mathbf{1} * (\sim x) = \sim (x/1) \leq (\sim x)$ and $\sim x = \mathbf{e} * (\sim x) \leq \mathbf{1} * (\sim x)$, we get that $\mathbf{1} * (\sim x) = \ \sim x$. Similarly, we get that $(\sim x) * \mathbf{1} = \ \sim x$. ■

It is then clear that we have a functor $\mathsf{N} : \mathrm{iRL_b} \to \mathrm{iIRL}$ and by restriction also a functor $\mathsf{N} : \mathrm{NRL} \to \mathrm{iIRL}$. But Lemma 7.9 allows us to exhibit also the following relation between $\widehat{(\_)}$ and $\mathsf{K}^\bullet$.

Proposition 7.11. *The diagram*

$$\begin{tikzcd} \mathrm{IRL}_0 \arrow[r, "\mathsf{K}^\bullet"] \arrow[d, "\widehat{(\_)}"'] & \mathrm{DRL} \arrow[d] \\ \mathrm{NRL} \arrow[r, "\mathsf{N}"'] & \mathrm{iIRL} \end{tikzcd}$$

commutes.

Remark 7.12. We can now give a more explicit description of the isomorphisms between the congruence lattices stated after Corollary 7.7.

Let $L \in \mathrm{IRL}_0$ and $\theta \in \mathrm{Con}L$. Then $\hat{\theta} := \theta \times \theta^{op} \in \mathrm{Con}\hat{L}$ and $\breve{\theta} := \hat{\theta} \cap (\mathsf{K}^\bullet L)^2 \in \mathrm{Con}(\mathsf{K}^\bullet L)$. It can be seen that the correspondence $\theta \mapsto \breve{\theta}$

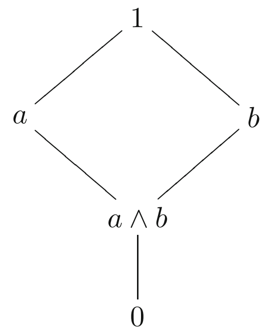
defines a lattice isomorphism between $\mathrm{Con}L$ and $\mathrm{Con}(\mathsf{K}^\bullet L)$. However, in general $\mathsf{K}^\bullet(L/\theta)$ is not isomorphic to $(\mathsf{K}^\bullet L)/\breve{\theta}$, as Example 7.13 shows.

On the other hand, for $T \in \mathrm{DRL}$ and $\alpha \in \mathrm{Con}T$, the correspondence $\alpha \mapsto \alpha_{\mathbf{c}} := \alpha \cap (\mathsf{C}T)^2$ defines a lattice morphism from $\mathrm{Con}T$ to $\mathrm{Con}(\mathsf{C}T)$. Take now $\phi \in \mathrm{Con}(\mathsf{C}T)$. Since $\varepsilon_T : T \to \mathsf{K}^\bullet\mathsf{C}T$ is a monomorphism, $\breve{\phi}$ induces a congruence in T. Hence, we have a bijection between $\mathrm{Con}T$ and $\mathrm{Con}(\mathsf{C}T)$, and, in this case,

$$(\mathsf{C}T)/\alpha_{\mathbf{c}} \cong \mathsf{C}(T/\alpha)$$

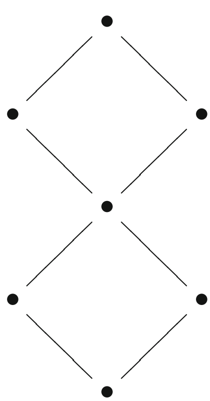
Example 7.13. Consider the following lattice with its Heyting algebra structure,

$L =$

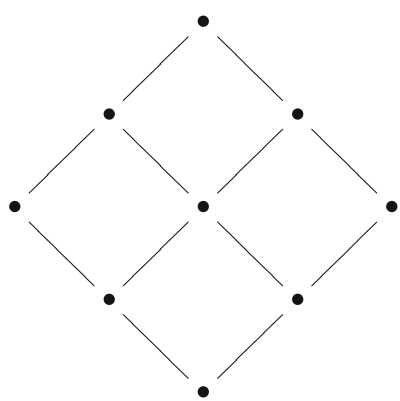


and θ given by the partition $\{\{0, a \wedge b\}, \{a\}, \{b\}, \{1\}\}$.

We have that $(\mathsf{K}^\bullet L)/\breve{\theta}$ and $\mathsf{K}^\bullet(L/\theta)$ are respectively isomorphic to



and



8. Left adjoints to the cones

Let $\mathcal{K}$ be a class of algebras, closed by taking subalgebras, and X a subset of an algebra $A \in \mathcal{K}$. In this section we write $\mathrm{Sub}_{\mathcal{K}}(X)$ for the $\mathcal{K}$-subalgebra of A generated by X.

Definition 8.1. Let $M \in$ cuRL. We define

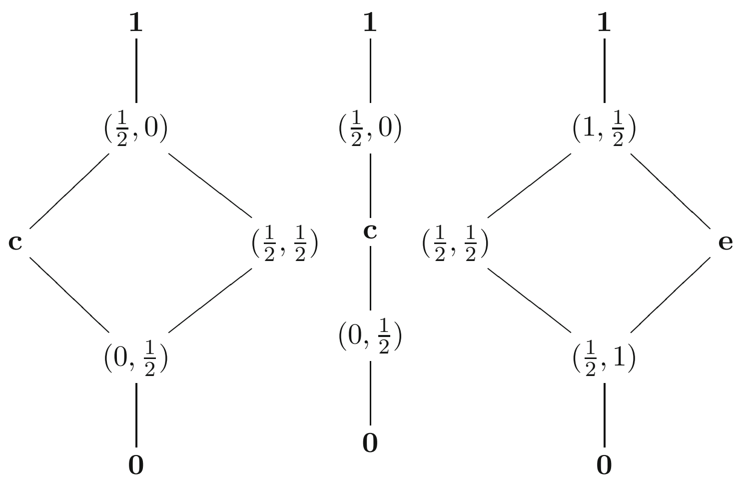
$$\mathsf{N}_{\mathbf{e}}(M) := \mathrm{Sub}_{\mathrm{iRL}}(\mathsf{C}_{\mathbf{e}}M) \leq M \tag{7}$$

Remark 8.2. Note that the subalgebra $\mathsf{N}_{\mathbf{e}}(M)$ makes sense for any involutive residuated lattice such that $\sim \mathbf{e} = \mathbf{e}$; i.e., for any $M \in \mathbf{e}$iRL. For instance, it makes sense for the result of applying the construction of Theorem 4.4 to any (not necessarily bounded below) integral residuated lattice.

Definition 8.3. Let $M \in$ NRL. We define

$$\mathsf{N}_{\mathbf{c}}(M) := \mathrm{Sub}_{\mathrm{iRL}}(\mathsf{C}M) \leq \mathsf{N}(M) \tag{8}$$

Example 8.4. Consider the MV-algebra with 3 elements $\{0, \frac{1}{2}, 1\}$, L_3. We have that $\mathsf{N}(\widehat{L}_3)$, $\mathsf{N}_{\mathbf{c}}(\widehat{L}_3)$ and $\mathsf{N}_{\mathbf{e}}(\widehat{L}_3)$ are



Let $L \in \mathrm{IRL}_0$. Let us write $\mathrm{M}^\bullet L := \mathsf{N}_{\mathbf{c}}(\widehat{L})$, and $M^\bullet : \mathrm{IRL}_0 \to \mathrm{DRL}$ for the only functor defined by this assignment on objects (Lemma A.1).

Theorem 8.5. *Functors* $M^\bullet : \mathrm{IRL}_0 \rightleftarrows \mathrm{DRL} : \mathsf{C}$ *form an adjoint pair.*

This theorem is a particular case of Proposition A.4.

Let $L \in \mathrm{IRL}$. Let us write $\mathrm{M}_\bullet L := \mathsf{N}_{\mathbf{e}}(\widehat{L})$, and $\mathrm{M}_\bullet : \mathrm{IRL} \to \mathbf{e}\mathrm{iRL}$ for the only functor defined by this assignment on objects (Lemma A.1). We have that,

Theorem 8.6. *Functors* $\mathrm{M}_\bullet : \mathrm{IRL} \rightleftarrows \mathbf{e}\mathrm{iRL} : \mathsf{C}_{\mathbf{e}}$ *form an adjoint pair.*

PROOF. For any $L \in \mathrm{IRL}$, let us consider the map $\eta_L : L \to \mathsf{C}_{\mathbf{e}}(\mathrm{M}_{\bullet}L)$ given by $\eta_L(a) := (a, 1)$, for $a \in L$.

For any $X \in \mathbf{e}\mathrm{iRL}$, consider the unique map $\varepsilon_X : \mathrm{M}_{\bullet}(\mathsf{C}_{\mathbf{e}}X) \to X$ in $\mathbf{e}$iRL, completely defined by $\varepsilon(x \wedge \mathbf{e}, \mathbf{e}) := (x \wedge \mathbf{e})$, for any $x \in X$.

A straightforward computation shows us that they are the unit and counit of the mentioned adjunction. ■

9. Generalized Nelson algebras

Let H be a Heyting algebra and write $T = \mathsf{K}^{\bullet}H$. We know, by Proposition 3.7, that T is a centered Nelson Algebra, whose Nelson implication is given by

$$(x_0, x_1) \Rightarrow (y_0, y_1) := (x_0 \to y_0, x_0 \wedge y_1) \tag{9}$$

In the original signature of Nelson algebras there is no product, although they are a class of algebras with involution and implication.

If $\Rightarrow$ were the residue in a commutative residuated lattice with involution, then the following identities should verify,

$$(x_0, x_1) \circ (y_0, y_1) = (y_0, y_1) \circ (x_0, x_1) \tag{10}$$

$$(x_0, x_1) \circ (y_0, y_1) = \sim [(x_0, x_1) \Rightarrow \sim (y_0, y_1)] = (x_0 \wedge y_0, x_0 \to y_1) \tag{11}$$

$$(y_0, y_1) \circ (x_0, x_1) = \sim [(y_0, y_1) \Rightarrow \sim (x_0, x_1)] = (x_0 \wedge y_0, y_0 \to x_1) \tag{12}$$

However, this is not generally true.

Thus, we can try with a "symmetrization" of the product, and see if we get back a new associative, commutative and residuated product,

$$\begin{aligned}(x_0, x_1) * (y_0, y_1) &:= [(x_0, x_1) \circ (y_0, y_1)] \vee [(y_0, y_1) \circ (x_0, x_1)] \\ &= (x_0 \wedge y_0, (x_0 \to y_1) \wedge (y_0 \to x_1))\end{aligned}$$

Note that this new product is exactly the one we obtain in $\mathsf{K}^{\bullet}L$, which we know satisfies all the desired conditions. Clearly, the residue of $*$ is not $\Rightarrow$, but the implication of $\mathsf{K}^{\bullet}L$.

Anyway, the Nelson structure on KL, $(\mathsf{K}^{\bullet}L, \wedge, \vee, \Rightarrow, , \mathbf{0}, \mathbf{1})$ is equationally equivalent to that of $\mathsf{K}^{\bullet}L$, $(\mathsf{K}^{\bullet}L, \wedge, \vee, *, \to, \sim, \mathbf{0}, \mathbf{1})$ (see [1] for reference) by the translations,

$$\begin{aligned}x * y &:= \sim (x \Rightarrow \sim y) \vee \sim (y \Rightarrow \sim x) \\ x \to y &:= (x \Rightarrow y) \wedge (y \Rightarrow x)\end{aligned}$$

and

$$x \Rightarrow y := (x * x) \to y$$

Hence, centered Nelson algebras and the variety we obtain by applying $\mathsf{K}^\bullet$ to Heyting algebras are equationally equivalent. These algebras are called *Nelson lattices* in [1], where it is shown that they form a quasivariety. They are algebraic models of constructive logic with strong negation.

For centered Nelson algebras, it is still possible to define an associative (although noncommutative) binary operation, left adjoint of the implication. More explicitly, let us define an associative product on a Nelson algebra $(N, \wedge, \vee, \Rightarrow, \neg, \mathbf{1}, \mathbf{0})$ by $x \star y := x * y * y$; then, we have that

$$x \leq y \Rightarrow z \text{ iff } x \leq y * y \to z \text{ iff } x * (y * y) \leq z \text{ iff } x \star y \leq z$$

This operation has a left unit ($e = \mathbf{1}$), but it fails to be a monoidal.

Let us call *generalized Nelson lattice*, (GNL) to a residuated Lattice which isomorphic to $\mathsf{K}^\bullet L$ for some integral residuated lattice with zero L.

As in the case of Nelson lattices, it can be shown that GNLs form a quasivariety. In a forthcoming work we study some particular subcategories of the category of GNLs.

A. Further left adjoints

Let $\mathcal{A}$ and $\mathcal{X}$ be categories of algebras and let $\mathsf{U}_\mathcal{A} : \mathcal{A} \to \mathbf{Sets}$ and $\mathsf{U}_\mathcal{X} : \mathcal{X} \to \mathbf{Sets}$ be the corresponding underlying set functors. (When there is no risk of confusion we will denote the forgetful functors simply by U.)

Let $H : \mathcal{A} \to \mathcal{X}$ be a functor and let $\tau : \mathsf{U}_\mathcal{A} \to \mathsf{U}_\mathcal{X} H$ be a natural transformation. In other words, inside the following (non necesarily commutative) diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[dr, "\mathsf{U}"'] & \mathcal{X} \arrow[d, "\mathsf{U}"] \\ & \mathbf{Sets} \end{tikzcd}$$

there is a 2-cell τ.

Now, for any A in $\mathcal{A}$ let FA in $\mathcal{X}$ be the subalgebra of HA generated by the elements of the form τa with $a \in A$.

Lemma A.1. *The assignment $A \mapsto FA$ extends to a functor $F : \mathcal{A} \to \mathcal{X}$ in such a way that the subalgebra inclusions $FA \to HA$ form a natural transformation $\iota : F \to H$. Moreover, the natural transformation τ factors through $\mathsf{U}_\mathcal{X}\iota$ inducing in this way a natural $\mathsf{U}_\mathcal{A} \to \mathsf{U}_\mathcal{X} F$ which we denote again by τ.*

Proof. Let $f : A \to A'$ be a morphism in $\mathcal{A}$. By standard results in universal algebra, FA consists of the elements of the form $t(\tau a_1, \ldots, \tau a_n)$

where t is a an n-ary term for some n in the signature of the presentation of $\mathcal{X}$ and $a_1, \dots, a_n$ are elements of A. Define $(Ff)(t(\tau a_1, \dots, \tau a_n)) = t(\tau(fa_1), \dots, \tau(fa_n))$. If it was well-defined then it is clearly a morphism. So suppose that s is an n-ary term (there is no loss of generality in assuming that the arity is the same) and $b_1, \dots, b_n$ are elements in A such that $s((\tau b_1), \dots, (\tau b_n)) = t((\tau a_1), \dots, (\tau a_n))$. Then $(Hf)s((\tau b_1), \dots, (\tau b_n)) = (Hf)t((\tau a_1), \dots, (\tau a_n))$. Since Hf is a morphism,

$$s((Hf)(\tau b_1), \dots, (Hf)(\tau b_n)) = t((Hf)(\tau a_1), \dots, (Hf)(\tau a_n))$$

Since τ is natural, $s(\tau(fb_1), \dots, \tau(fb_n)) = t(\tau(fa_1), \dots, \tau(fa_n))$.

Finally, the definition of Ff clearly shows that we have a commutative diagram as below

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Ff"'] & HA \arrow[d, "Hf"] \\ FA' \arrow[r] & HL' \end{tikzcd}$$

in $\mathcal{X}$ where the horizontal maps are the subalgebra inclusions. The last part of the result follows from the definition of F. ∎

Let us now consider a more restricted situation. Let $H : \mathcal{A} \to \mathcal{H}$ be a full and faithful functor with a left adjoint G. Assume also that there is a natural transformation $\kappa : \mathsf{U}_{\mathcal{A}} G \to \mathsf{U}_{\mathcal{X}}$. Since H is full and faithful the counit ε is an iso. So κ induces another transformation as follows

$$\mathsf{U}_{\mathcal{A}} \xrightarrow{\mathsf{U}\varepsilon^{-1}} \mathsf{U}_{\mathcal{A}} GHA \xrightarrow{\kappa_{HA}} \mathsf{U}_{\mathcal{X}} HA$$

and we denote it by $\tau : \mathsf{U}_{\mathcal{A}} \to \mathsf{U}_{\mathcal{X}} H$. By Lemma A.1, this τ induces a functor $F : \mathcal{A} \to \mathcal{X}$.

If $\eta : Id \to HG$ is the unit of the adjunction let us say that a natural transformation $\kappa : \mathsf{U}_{\mathcal{A}} G \to \mathsf{U}_{\mathcal{X}}$ as above is *conical* if the following diagram

$$\begin{tikzcd} \mathsf{U}_{\mathcal{A}} GHG \arrow[r, "\mathsf{U}_{\mathcal{A}} \varepsilon_G"] \arrow[rrd, "\kappa_{HG}"'] & \mathsf{U}_{\mathcal{A}} G \arrow[r, "\kappa"] & \mathsf{U}_{\mathcal{X}} \arrow[d, "\mathsf{U}_{\mathcal{X}} \eta"] \\ & & \mathsf{U}_{\mathcal{X}} HG \end{tikzcd}$$

commutes.

LEMMA A.2. *In the situation just presented, if κ is a conical transformation then the following diagram*

$$\begin{tikzcd} \mathsf{U}_{\mathcal{A}}G \arrow[r, "\kappa"] \arrow[dr, "\tau_G"'] & \mathsf{U}_{\mathcal{X}} \arrow[d, "\mathsf{U}_{\mathcal{X}}\eta"] \\ & \mathsf{U}_{\mathcal{X}}HG \end{tikzcd}$$

commutes.

PROOF. This is mainly a matter of notation. Just precompose the definition of conical transformation with $\mathsf{U}_{\mathcal{A}}\varepsilon_G^{-1} : \mathsf{U}_{\mathcal{A}}G \to \mathsf{U}_{\mathcal{A}}GHG$ and compare with the definition of τ_G. ∎

Another useful fact is the following.

LEMMA A.3. *If in the same situation of Lemma A.2, $G\iota : GF \to GH$ is an iso, then the following diagram*

$$\begin{tikzcd} \mathsf{U}_{\mathcal{A}} \arrow[r, "\varepsilon^{-1}"] \arrow[drr, "\tau_A"'] & \mathsf{U}_{\mathcal{A}}GH \arrow[r, "(G\iota)^{-1}"] & \mathsf{U}_{\mathcal{A}}GF \arrow[d, "\kappa_F"] \\ & & \mathsf{U}_{\mathcal{X}}F \end{tikzcd}$$

commutes.

PROOF. Just postcompose with $\mathsf{U}_{\mathcal{X}}\iota$ and calculate. ∎

PROPOSITION A.4. *Let $H : \mathcal{A} \to \mathcal{X}$ be full and faithful and have a faithful left adjoint $G : \mathcal{X} \to \mathcal{A}$. Let $\kappa : \mathsf{U}_{\mathcal{A}}G \to \mathsf{U}_{\mathcal{X}}$ be a conical natural transformation and let τ and $F : \mathcal{A} \to \mathcal{X}$ be the induced natural transformation and functor. If $\kappa : \mathsf{U}_{\mathcal{A}}G \to \mathsf{U}_{\mathcal{X}}$ is mono and $G\iota : GF \to GH$ is an iso then F is left adjoint to G.*

PROOF. Denote the composition of $(G\iota)^{-1}\varepsilon^{-1}$ by $\nu : Id \to GF$. We show that this natural transformation is the unit of the adjunction.

Let $f : A \to GX$. We extend f to a function $\overline{f} : FA \to X$ as follows: $\overline{f}t(\tau a_1, \ldots, \tau a_n) = t(\kappa(fa_1), \ldots, \kappa(fa_n))$. Notice that we have as a particular case that $(\mathsf{U}\overline{f})\tau = \kappa(\mathsf{U}f)$. (Here we are using the $\tau : \mathsf{U}_{\mathcal{A}} \to \mathsf{U}_{\mathcal{X}}F$.)

To prove that it is well defined, suppose that s is another n-ary term and $b_1, \ldots, b_n$ are elements in A such that

$$s((\tau b_1), \ldots, (\tau b_n)) = t((\tau a_1), \ldots, (\tau a_n)).$$

Since G is faithful, the unit $\eta : X \to HGX$ is mono so from the following calculation

$$(\mathsf{U}\eta)s(\kappa(fb_1),\ldots,\kappa(fb_n)) = s((\mathsf{U}\eta)(\kappa(fb_1)),\ldots,(\mathsf{U}\eta)(\kappa(fb_n))) =$$
$$= s(\tau(fb_1),\ldots,\tau(fb_n)) = s((\mathsf{U}Hf)(\tau b_1),\ldots,(\mathsf{U}Hf)(\tau b_n)) =$$
$$= (\mathsf{U}Hf)s(\tau b_1,\ldots,\tau b_n) = (\mathsf{U}Hf)t(\tau a_1,\ldots,\tau a_n) = \ldots =$$
$$= \ldots = (\mathsf{U}\eta)t(\kappa(fa_1),\ldots,\kappa(fa_n))$$

we can conclude that $s(fb_1,\ldots,fb_n) = t(fa_1,\ldots,fa_n)$.

We must now prove that $(G\overline{f})\nu = f$. Since $\mathsf{U}_{\mathcal{A}}$ is faithful, it is enough to prove that $(\mathsf{U}G\overline{f})(\mathsf{U}\nu) = \mathsf{U}f$. We now use that κ is mono together with Lemma A.3 and calculate:

$$\kappa_X(\mathsf{U}G\overline{f})(\mathsf{U}\nu) = (\mathsf{U}\overline{f})\kappa_{FA}(\mathsf{U}((G\iota)^{-1}))(\mathsf{U}\varepsilon^{-1}) = (\mathsf{U}\overline{f})\tau_A = \kappa_X(\mathsf{U}f)$$

so the result follows. ■

B. Congruences

Let X be an object in a category. The set of regular epis with domain X can be pre-ordered by stating that $(q : X \to Q) \leq (q' : X \to Q')$ if there exists a map $f : Q \to Q'$ such that $fq = f'$. The pre-order induces an equivalence relation and we denote the resulting quotient by $\mathbf{Qt}(X)$.

Varieties are exact (as categories) so taking coequalizers induces a natural isomorphism between the set of congruences on X and the set $\mathbf{Qt}(X)$. In this context $\mathbf{Qt}(X)$ extends to a contravariant functor (with action given by pullback of congruences).

Proposition B.1. *Let $\mathcal{C}$ have pushouts along regular epis and let $L : \mathcal{C} \to \mathcal{D}$ be faithful and have both left and right adjoints $M \dashv L \dashv K$. If M (or equivalently K) is full and faithful then the induced $\mathbf{Qt}(X) \to \mathbf{Qt}(LX)$ is an isomorphism for every X in $\mathcal{C}$. Moreover, so is $\mathbf{Qt}(A) \to \mathbf{Qt}(KA)$.*

Applying Proposition B.1 to the adjunction $\mathsf{M} \dashv \mathsf{L} \dashv \mathsf{K} : \mathrm{DL} \to \mathrm{Kl}_c$ presented in Theorem 3.2 we obtain the restriction of Corollary 1.10 in [3] to centered Kleene algebras, namely: that for each centered Kleene algebra A, $\mathbf{Qt}A \cong \mathbf{Qt}(\mathsf{L}A)$.

Lemma B.2. *Let $\mathcal{C}$ be a category with pushouts along regular epis. Let $L : \mathcal{C} \to \mathcal{D}$ preserve regular epis and so, induce a function $\mathbf{Qt}(X) \to \mathbf{Qt}(LX)$. If L preserves pushouts along regular epis and has a full and faithful left adjoint then the function $\mathbf{Qt}(X) \to \mathbf{Qt}(LX)$ is surjective.*

PROOF. Let $M : \mathcal{D} \to \mathcal{C}$ be the left adjoint to L and let $\varepsilon_X : MLX \to X$ be the counit of the adjunction. The unit $\eta_A : A \to LMA$ is an iso because M is full and faithful.

Let $q : LX \to Q$ be regular epi. As M is left adjoint it preserves regular epis so Mq is one. Take the pushout along the counit as on the left below

$$\begin{tikzcd} MLX \arrow[r, "\varepsilon"] \arrow[d, "Mq"'] & X \arrow[d, "p_1"] \\ MQ \arrow[r, "p_0"'] & R \end{tikzcd} \qquad \begin{tikzcd} LMLX \arrow[r, "L\varepsilon_X"] \arrow[d, "LMq"'] & LX \arrow[d, "Lp_1"] \\ LMQ \arrow[r, "Lp_0"'] & LR \end{tikzcd}$$

and as L preserves pushouts along regular epis, the square on the right above is a pushout. But $L\varepsilon_x$ is an iso (indeed, the inverse of η_{LX}). So its pushout Lp_0 is an iso. Attaching the naturality square induced by q and η to the left of the right diagram above shows that Lp_1 and q induce the same element in $\mathbf{Qt}(LX)$. So the function $\mathbf{Qt}(X) \to \mathbf{Qt}(LX)$ is surjective. ■

LEMMA B.3. *If $L : \mathcal{C} \to \mathcal{D}$ is faithful and it has a right adjoint then the induced $\mathbf{Qt}(X) \to \mathbf{Qt}(LX)$ is injective.*

PROOF. It is enough to show that for regular epis $q : X \to Q$ and $q' : X \to Q'$ with common domain in $\mathcal{C}$, $Lq \leq Lq'$ implies $q \leq q'$. So let $f : LQ \to LQ'$ be such that $f(Lq) = Lq'$.

Now let $e_0, e_1 : E \to X$ be the kernel pair of $X \to Q$. We need to check that $q'e_0 = q'e_1$. But we can calculate

$$(Lq')(Le_0) = f(Lq)(Le_0) = f(Lq)(Le_1) = (Lq')(Le_1)$$

so $L(q'e_0) = L(q'e_1)$ and as L is faithful $q'e_0 = q'e_1$. So the result follows. ■

PROOF OF PROPOSITION B.1. To prove that $\mathbf{Qt}(X) \to \mathbf{Qt}(LX)$ is an isomorphism just use the two lemmas above. To prove that $\mathbf{Qt}(A) \to \mathbf{Qt}(KA)$ is an iso let $X = KA$ and use that LKA is iso to A. ■

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