The Intensional Lambda Calculus*

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Abstract. We introduce a natural deduction formulation for the Logic of Proofs, a refinement of modal logic **S4** in which the assertion $\Box A$ is replaced by $[\![s]\!]A$ whose intended reading is "*s is a proof of A*". A term calculus for this formulation yields a typed lambda calculus $\lambda^{\mathbf{I}}$ that internalises *intensional* information on *how* a term is computed. In the same way that the Logic of Proofs internalises its own *derivations*, $\lambda^{\mathbf{I}}$ internalises its own *computations*. Confluence and strong normalisation of $\lambda^{\mathbf{I}}$ is proved. This system serves as the basis for the study of type theories that internalise intensional aspects of computation.

1 Introduction

This paper introduces a typed lambda calculus that internalises its own computations. Such a system is obtained by a propositions-as-types [GLT89] interpretation of a logical system for provability which internalises its own proofs, namely the Logic of Proofs **LP** [Art95,Art01]. Proofs are represented as combinatory terms known as *proof polynomials*. In the minimal propositional logic fragment of **LP** proof polynomials are constructed from proof variables and constants using two operations: application "$\cdot$" and proof-checker "!". The usual propositional connectives are augmented by a new one: given a proof polynomial s and a proposition A build $[\![s]\!]A$. The intended reading is: "*s is a proof of A*". The axioms and inference schemes of **LP** are:

A0. Axiom schemes of minimal logic in the language of **LP**	
A1. $[\![s]\!]A \supset A$	*"verification"*
A2. $[\![s]\!](A \supset B) \supset ([\![t]\!]A \supset [\![s \cdot t]\!]B)$	*"application"*
A3. $[\![s]\!]A \supset [\![!s]\!][\![s]\!]A$	*"proof checker"*
R1. $\Gamma \vdash A \supset B$ and $\Gamma \vdash A$ implies $\Gamma \vdash B$	*"modus ponens"*
R2. If $\mathbf{A}$ is an axiom **A0**-**A3**, and c is a proof constant, then $\vdash [\![c]\!]\mathbf{A}$	*"necessitation"*

For verification one reads: "*if s is a proof of A, then A holds*". As regards the proof polynomials the standard interpretation is as follows. For application one reads: "*if s is a proof of $A \supset B$ and t is a proof of A, then $s \cdot t$ is a proof of B*". Thus "$\cdot$" represents composition of proofs. For proof checking one reads: "*if s is a proof of A, then $!s$ is a proof of the sentence* 's is a proof of A' ". Thus $!s$ is seen as a computation that verifies $[\![s]\!]A$.

* Extended report of LFCS07 submission. Version of: December 28, 2006

First we introduce a natural deduction (ND) formulation $\mathbf{LP}^{-}_{nd}$ for **LP**. Following recent work on *judgemental reconstruction* [ML83] of intuitionistic **S4** [DP96,DP01b,DP01a], judgements are introduced in which a distinction is made between propositions whose *truth* is assumed from those whose *validity* is assumed. Judgements in $\mathbf{LP}^{-}_{nd}$ are of the form:

$$v_1 : A_1 \ \mathit{valid}, \ldots, v_n : A_n \ \mathit{valid}; a_1 : B_1 \ \mathit{true}, \ldots, a_m : B_m \ \mathit{true} \vdash A \ \mathit{true} \mid s$$

which expresses "*s is evidence that A is true, assuming that for each $i \in 1..n$, v_i is evidence that A_i is valid and assuming that for each $j \in 1..m$, a_j is evidence that B_j is true*". Such judgements are called hypothetical judgements [ML83]. Evidence s is a constituent part of the judgement without which the proposed reading is no longer possible. Its importance is reflected in the following introduction rule for the $[\![s]\!]$ connective:

$$\frac{\Delta; \cdot \vdash A \mid s}{\Delta; \Gamma \vdash [\![s]\!]A \mid !s} \Box\mathsf{I}$$

This scheme internalises proofs of validity: If s is evidence that A is unconditionally true ("$\cdot$" indicates an empty set of hypothesis of truth), then it is true that s is a proof of A. The new witness to this fact is registered as the evidence $!s$. The "!" operator is reminiscent of that of proof polynomials. However, in $\mathbf{LP}^{-}_{nd}$, proof terms such as s encode ND derivations and thus are no longer the proof polynomials of **LP**.

At the basis of the meaning of hypothetical judgements (provided by the axioms and inference schemes presented in Sec. 2) is the notion of substitution. The following two principles, the Substitution Principle for Truth with Evidence and the Substitution Principle for Validity with Evidence, reflect the true hypothetical nature of hypothesis.

- If $\Delta; \Gamma \vdash A \mid s$ and $\Delta; \Gamma, a : A, \Gamma' \vdash B \mid t$, then $\Delta; \Gamma, \Gamma' \vdash B \mid t^{a}_{s}$
- If $\Delta; \cdot \vdash A \mid s$ and $\Delta, v : A, \Delta'; \Gamma \vdash B \mid t$, then $\Delta, \Delta'; \Gamma \vdash B^{v}_{s} \mid t^{v}_{s}$

These principles allow derivations to be composed, a fundamental operation on which the process of normalisation of derivations relies on. In fact, composition of derivations suffices, in general, to formulate rules for eliminating redundancy in derivations. However, the fact that $\mathbf{LP}^{-}_{nd}$ internalises its own proofs presents a complication in this respect. For example, the naïve normalisation step depicted in Fig. 1 which relies on the Substitution Principle for Truth with Evidence fails given that it modifies the judgement that was originally justified. On a more pragmatical level, such a normalisation process may produce invalid derivations. Consider the following derivation

$$\dfrac{\dfrac{\dfrac{v : A; a : A \vdash A \mid a}{v : A; \cdot \vdash A \supset A \mid \lambda a : A.a} \supset\mathsf{I} \qquad v : A; \cdot \vdash A \mid v}{v : A; \cdot \vdash A \mid (\lambda a : A.a) \cdot v} \supset\mathsf{E}}{v : A; \cdot \vdash [\![(\lambda a : A.a) \cdot v]\!]A \mid !((\lambda a : A.a) \cdot v)} \Box\mathsf{I}$$

$$\dfrac{\dfrac{\Delta; \Gamma, a : A \vdash B \mid s}{\Delta; \Gamma \vdash A \supset B \mid \lambda a : A.s} \supset \mathsf{I} \qquad \Delta; \Gamma \vdash A \mid t}{\Delta; \Gamma \vdash B \mid (\lambda a : A.s) \cdot t} \supset \mathsf{E} \quad \rightsquigarrow \quad \Delta; \Gamma \vdash B \mid s^a_t$$

Fig. 1. Naïve simplification

If the normalisation step of Fig. 1 were applied to the subderivation ending in the judgement $v : A; \cdot \vdash A \mid (\lambda a : A.a) \cdot v$, then the application of $\Box\mathsf{I}$ in the resulting derivation would not be valid.

The problem stems in that the normalisation step is attempting to identify, at the meta-level, the two derivations and $\mathbf{LP}^-_{nd}$ happens to internalise its own derivations. As a consequence, the normalisation step must be reflected in the logic too. More precisely, a new judgement expressing the equality on evidence must be introduced. Accordingly, in Sec. 2.2 we extend our ND presentation $\mathbf{LP}^-_{nd}$ with *hypothetical judgements for evidence equality.* The normalisation process is thus internalised into the logic. For this amended system, $\mathbf{LP}_{nd}$, the set of derivations is seen to be closed under normalisation.

In Sec. 3 we study a term assignment for $\mathbf{LP}_{nd}$, namely the *intensional lambda calculus* ($\lambda^{\mathbf{I}}$). $\lambda^{\mathbf{I}}$ results from extending the propositions-as-types correspondence to $\mathbf{LP}_{nd}$. The normalisation process of derivations in $\mathbf{LP}_{nd}$ yields a notion of *reduction* on the typed lambda calculus terms. Just as $\mathbf{LP}_{nd}$ internalises its own derivations, the operational counterpart of this logic is seen to internalise the reduction of derivations. We show that $\lambda^{\mathbf{I}}$ is strongly normalising and confluent by applying properties of higher-order rewrite systems.

Related work. S. Artemov introduced the Logic of Proofs in [Art95,Art01]. A ND presentation for **LP** is provided in [Art01]. This presentation relies on combinatory terms as proof terms (proof polynomials). It is a ND system for a logic that internalises Hilbert style proofs. As a consequence, the presence of normalisation is not felt at the level of proof terms. Since we use proof terms that encode ND proofs, the internalisation scheme implemented by $\Box\mathsf{I}$ together with the normalisation process on derivations has a visible impact in the design of the inference schemes for our system $\mathbf{LP}_{nd}$.

V. Brezhnev [Bre01] formulates a system of labeled sequents. Roughly, a refinement of the sequent presentation of **LP** [Art01] is presented in which labeled sequents are derived rather than the sequents themselves. It has been proved [Art95,Art01] that **LP** is a refinement of **S4** in the sense that any cut-free derivation of **S4** can be *realized* by one of **LP**. A realization of an **S4** derivation is the process of appropriately filling in all occurrences of boxes $\Box$ with proof polynomials such that a valid **LP** derivation is obtained. The aim of the work of Brezhnev is to make this correspondence explicit. Also, he extends the correspondence to other modal logics such as **K**, **K4**, **D**, **D4** and **T**.

From a type theoretic perspective we should mention the theory of dependent types [Bar92]. Dependent type theory is the type-theoretic counterpart of first-order logic via the propositions-as-types correspondence. Types may depend on terms, in much the same way that a type $[\![s]\!]A$ depends on the proof term s. In contrast to $\lambda^{\mathbf{I}}$, dependent type theory lacks a notion of internalisation of derivations.

More closely related to $\lambda^{\mathbf{I}}$ is the reflective λ-calculus (λ^{∞}) [AA01]. λ^{∞} is a rigidly typed (all variables and subterms carry a fixed type) lambda calculus which essentially results from a term assignment of the aforementioned ND presentation of [Art01]. The difference with the approach of this paper is that in the reflective λ-calculus $[\![s]\!]A$ is read as "*s has type A*". Accordingly, hypothesis are not labeled with variables, rather they are part of the formula. For example, $x : A \vdash x : A$ becomes $[\![x]\!]A \vdash [\![x]\!]A$. An unwanted complication is that the desired internalisation property (namely, $A_1, A_2, \ldots, A_n \vdash B$ implies that for fresh variables $x_1, x_2, \ldots, x_n$ there exists a term $t(x_1, x_2, \ldots, x_n)$ such that $[\![x_1]\!]A_1, [\![x_2]\!]A_2, \ldots, [\![x_n]\!]A_n \vdash [\![t(x_1, x_2, \ldots, x_n)]\!]B$) changes the types of the assumptions. As a consequence, operations on types having nested copies of proof terms are required for typing. This also complicates the definition of reduction on terms.

2 Natural Deduction for LP

Following [DP01b] we distinguish the following judgements: "*A is a proposition*" ("*A proposition*" for short), "*A true*" and "*A valid*". In the case of the second and third judgements we assume that it is already known that "*A proposition*". The inference schemes defining the meaning of "*A proposition*" are the usual well-formedness conditions and hence are omitted. For example, in the case of "*$A \supset B$ proposition*" we have the inference scheme:

$$\frac{A \textit{ proposition} \quad B \textit{ proposition}}{A \supset B \textit{ proposition}}$$

Our interest lies in providing meaning to the following *hypothetical judgements with explicit evidence*:

$$v_1 : A_1 \textit{ valid}, \ldots, v_n : A_n \textit{ valid}; a_1 : B_1 \textit{ true}, \ldots, a_m : B_m \textit{ true} \vdash A \textit{ true} \mid s$$

by a set of axiom schemes and inference schemes, where v_i, $i \in 1..n$, and a_j, $j \in 1..m$, range over some given some set of *evidence (of proof) variables* $\{x_1, x_2, \ldots\}$. To the left of the semi-colon we place the assumptions of validity and to the right the assumptions of truth. For the sake of readability, we drop the qualifiers "*valid*" and "*true*". Consequently, these judgements take the form:

$$v_1 : A_1, \ldots, v_n : A_n; a_1 : B_1, \ldots, a_m : B_m \vdash A \mid s$$

In addition to the usual requirement that the v_i and a_i be distinct, we must also require that they be fresh (i.e. that they do not occur in the A_i and B_i). We refer to this as the labeling condition. Also, since we assume J_1 through J_n, in a hypothetical proof of a hypothetical judgement with explicit evidence, we may use the J_i as if we knew them. As a consequence we can substitute an arbitrary derivation of J_i for all its uses by means of the two aforementioned substitution principles. Once we have established the meaning of hypothetical judgements with explicit evidence we shall in fact be able to prove these principles.

2.1 Axiom and Inference Schemes

It is convenient to introduce first a preliminary ND system ($\mathbf{LP}^{-}_{nd}$), point out its weaknesses and then introduce the final ND system $\mathbf{LP}_{nd}$. We begin by defining the set of Proof Terms, Propositions, Truth Contexts and Validity Contexts.

$$\begin{array}{ll}
\textit{Proof Terms} & s ::= x \mid s \cdot s \mid \lambda a : A.s \mid !s \mid \text{X}\textsc{trt}\, s\, \textsc{as}\, v : A\, \textsc{in}\, s \\
\textit{Propositions} & A ::= P \mid A \supset A \mid [\![s]\!]A \\
\textit{Truth Contexts} & \Gamma ::= \cdot \mid \Gamma, a : A \\
\textit{Validity Contexts} & \Delta ::= \cdot \mid \Delta, v : A
\end{array}$$

We write $\mathsf{fv}(s)$ for the set of free variables of a proof term. All free occurrences of a (resp. v) in s are bound in $\lambda a : A.s$ (resp. $\text{X}\textsc{trt}\, t\, \textsc{as}\, v : A\, \textsc{in}\, s$). A proposition is either a propositional variable P, an implication $A \supset B$ or a validity proposition $[\![s]\!]A$. Truth and validity contexts are sequences of labeled propositions; "$\cdot$" denotes the empty context. We write s^x_t for the result of substituting all free occurrences of x in s by t and assume that bound variables are renamed whenever necessary; likewise for A^x_t.

Definition 1. $\mathbf{LP}^{-}_{nd}$ *is defined by the schemes of Fig. 2.*

An informal explanation of some of these schemes follows. The axiom scheme oVar states that the judgement "$\Delta; \Gamma, a : A, \Gamma' \vdash A \mid a$" is evident in itself. Indeed, if we assume that a is evidence that proposition A is true, then we may immediately conclude that A is true with evidence a. Likewise, the mVar inference scheme states that if we assume that v is evidence that proposition A is valid, then we may use this evidence to immediately conclude that A is true. The inference schemes $\supset \mathsf{I}$ and $\supset \mathsf{E}$ are standard. The introduction scheme for the $[\![s]\!]$ modality internalises metalevel evidence into the object logic. It states that if s is unconditional evidence that A is true, then A is in fact valid with witness s (i.e. $[\![s]\!]A$ is true). Evidence for the truth of $[\![s]\!]A$ is constructed from the (verified) evidence that A is unconditionally true by prefixing it with a bang constructor. Finally, $\Box\mathsf{E}$ allows the discharging of validity hypothesis. In order to discharge the validity hypothesis $v : A$, a proof of the validity of A is required. In our system, this requires proving that $[\![r]\!]A$ is true with evidence s, for some evidence of proof r and s. Note that r is evidence that A is unconditionally true (i.e. valid) whereas s is evidence that $[\![r]\!]A$ is true. The former is then substituted in the place of all free occurrences of v in the proposition C. This construction is recorded with evidence $\text{X}\textsc{trt}\, s\, \textsc{as}\, v : A\, \textsc{in}\, t$ in the conclusion. The mnemonic symbols "Xtrt" stand for "extract" since, intuitively, evidence of the validity of A may be seen to be extracted from evidence of the truth of $[\![r]\!]A$. Two examples of derivations in $\mathbf{LP}^{-}_{nd}$ follow. The first one proves $[\![s]\!]A \supset A$.

Minimal Propositional Logic Fragment

$$\frac{}{\Delta;\Gamma, a:A, \Gamma' \vdash A \mid a}\ \mathsf{oVar}$$

$$\frac{\Delta;\Gamma, a:A \vdash B \mid s}{\Delta;\Gamma \vdash A\supset B \mid \lambda a:A.s}\ \supset \mathsf{I} \qquad \frac{\Delta;\Gamma \vdash A\supset B \mid s \quad \Delta;\Gamma \vdash A \mid t}{\Delta;\Gamma \vdash B \mid s\cdot t}\ \supset \mathsf{E}$$

Provability Fragment

$$\frac{}{\Delta, v:A, \Delta';\Gamma \vdash A \mid v}\ \mathsf{mVar}$$

$$\frac{\Delta;\cdot \vdash A \mid s}{\Delta;\Gamma \vdash [\![s]\!]A \mid !s}\ \Box\mathsf{I} \qquad \frac{\Delta;\Gamma \vdash [\![r]\!]A \mid s \quad \Delta, v:A;\Gamma \vdash C \mid t}{\Delta;\Gamma \vdash C^v_r \mid \textsc{Xtrt}\, s\, \textsc{as}\, v:A\, \textsc{in}\, t}\ \Box\mathsf{E}$$

Fig. 2. Explanation for Hypothetical Judgements with Explicit Evidence

$$\dfrac{\dfrac{\dfrac{}{\cdot; a:[\![s]\!]A \vdash [\![s]\!]A \mid a}\ \mathsf{oVar} \qquad \dfrac{}{v:A; a:[\![s]\!]A \vdash A \mid v}\ \mathsf{mVar}}{\cdot; a:[\![s]\!]A \vdash A \mid \textsc{Xtrt}\, a\, \textsc{as}\, v:A\, \textsc{in}\, v}\ \Box\mathsf{E}}{\cdot;\cdot \vdash [\![s]\!]A \supset A \mid \lambda a:[\![s]\!]A.\textsc{Xtrt}\, a\, \textsc{as}\, v:A\, \textsc{in}\, v}\ \supset\mathsf{I}$$

The second example proves $[\![s]\!]A\supset[\![!s]\!][\![s]\!]A$.

$$\dfrac{\dfrac{\dfrac{}{\cdot; a:[\![s]\!]A \vdash [\![s]\!]A \mid a}\ \mathsf{oVar} \qquad \dfrac{\dfrac{\dfrac{}{w:A;\cdot \vdash A \mid w}\ \mathsf{mVar}}{w:A;\cdot \vdash [\![w]\!]A \mid !w}\ \Box\mathsf{I}}{w:A; a:[\![s]\!]A \vdash [\![!w]\!][\![w]\!]A \mid !!w}\ \Box\mathsf{I}}{\cdot; a:[\![s]\!]A \vdash [\![!s]\!][\![s]\!]A \mid \textsc{Xtrt}\, a\, \textsc{as}\, w:A\, \textsc{in}\, !!w}\ \Box\mathsf{E}}{\cdot;\cdot \vdash [\![s]\!]A \supset [\![!s]\!][\![s]\!]A \mid \lambda a:[\![s]\!]A.\textsc{Xtrt}\, a\, \textsc{as}\, w:A\, \textsc{in}\, !!w}\ \supset\mathsf{I}$$

The standard structural properties of judgements (exchange, weakening and contraction) hold. Also, the substitution principles for truth with evidence and validity with evidence may be proved by induction on the derivation.

Lemma 1. *Some Properties of Judgements in* $\mathbf{LP}^-_{nd}$

1. *(Exchange) If* $\Delta, u:A, v:B, \Delta';\Gamma \vdash C \mid s$*, then* $\Delta, v:B, u:A, \Delta';\Gamma \vdash C \mid s$.
2. *(Exchange) If* $\Delta;\Gamma, a:A, b:B, \Gamma' \vdash C \mid s$*, then* $\Delta;\Gamma, b:B, a:A, \Gamma' \vdash C \mid s$.
3. *(Weakening) If* $\Delta, \Delta';\Gamma \vdash A \mid s$*, then* $\Delta, v:B, \Delta';\Gamma \vdash A \mid s$.
4. *(Weakening) If* $\Delta;\Gamma, \Gamma' \vdash A \mid s$*, then* $\Delta;\Gamma, a:B, \Gamma' \vdash A \mid s$.
5. *(Contraction) If* $\Delta, u:A, v:A, \Delta';\Gamma \vdash A \mid s$*, then* $\Delta, w:A, \Delta';\Gamma \vdash A^{u,v}_w \mid s^{u,v}_w$ *for* w *fresh.*

6. *(Contraction) If* $\Delta; \Gamma, a : A, b : A, \Gamma' \vdash A \mid s$, *then* $\Delta; \Gamma, c : A, \Gamma' \vdash A \mid s^{a,b}_c$ *for* c *fresh.*
7. *If* $\Delta; \Gamma \vdash A \mid s$ *and* $\Delta; \Gamma, a : A, \Gamma' \vdash B \mid t$, *then* $\Delta; \Gamma, \Gamma' \vdash B \mid t^a_s$.
8. *If* $\Delta; \cdot \vdash A \mid s$ *and* $\Delta, v : A, \Delta'; \Gamma \vdash B \mid t$, *then* $\Delta, \Delta'; \Gamma \vdash B^v_s \mid t^v_s$.

A more interesting property is that $\mathbf{LP}^-_{nd}$ internalises its own proofs of unconditional truth.

Lemma 2 (Lifting [Art95]). *Let* $\Delta = u_1 : A_1, \ldots, u_n : A_n$ *and* $\Gamma = b_1 : B_1, \ldots, b_m : B_m$. *If* $\Delta; \Gamma \vdash A \mid r$, *then* $\Delta, v_1 : B_1, \ldots, v_m : B_m; \cdot \vdash [\![s(\boldsymbol{u}, \boldsymbol{v})]\!]A \mid t(\boldsymbol{u}, \boldsymbol{v})$ *where* $s(\boldsymbol{u}, \boldsymbol{v}) = (\lambda \boldsymbol{b} : \boldsymbol{B}.r) \cdot v_1 \cdot v_2 \cdot \ldots \cdot v_m$ *and* $t(\boldsymbol{u}, \boldsymbol{v}) = \text{X}\textsc{trt}\, !\lambda \boldsymbol{b} : \boldsymbol{B}.r\, \textsc{as}\, u : (\boldsymbol{B} \supset A)\, \textsc{in}\, !(u \cdot v_1 \cdot v_2 \ldots \cdot v_m)$.

Proof. Let $\Delta; \cdot \vdash [\![\lambda \boldsymbol{b} : \boldsymbol{B}.r]\!](\boldsymbol{B} \supset A) \mid !\lambda \boldsymbol{b} : \boldsymbol{B}.r$ be the judgement obtained from $\Delta; \Gamma \vdash A \mid r$ by passing all truth assumptions from the left of the turnstile to the right of the turnstile using $\supset \mathsf{I}$ and then applying $\Box\mathsf{I}$ once. If Γ is empty, then we conclude by taking $s(\boldsymbol{u}, \boldsymbol{v}) = \lambda \boldsymbol{b} : \boldsymbol{B}.r = r$ and $t(\boldsymbol{u}, \boldsymbol{v}) = !\lambda \boldsymbol{b} : \boldsymbol{B}.r = !r$. Otherwise, by weakening we may further obtain a derivation of

$$\Delta, v_1 : B_1, \ldots, v_m : B_m; \cdot \vdash [\![\lambda \boldsymbol{b} : \boldsymbol{B}.r]\!](\boldsymbol{B} \supset A) \mid !\lambda \boldsymbol{b} : \boldsymbol{B}.r \tag{1}$$

Note also that the judgement

$$\Delta, v_1 : B_1, \ldots, v_m : B_m, u : (\boldsymbol{B} \supset A); \cdot \vdash [\![u \cdot v_1 \cdot v_2 \ldots \cdot v_m]\!]A \mid !(u \cdot v_1 \cdot v_2 \ldots \cdot v_m)$$

is derivable. Thus we may conclude with an application of $\Box\mathsf{E}$ and deduce that

$$\begin{aligned} s(\boldsymbol{u}, \boldsymbol{v}) &= (\ldots((((\lambda \boldsymbol{b} : \boldsymbol{B}.r) \cdot v_1) \cdot v_2) \cdot \ldots \cdot v_m \\ t(\boldsymbol{u}, \boldsymbol{v}) &= \text{X}\textsc{trt}\, !\lambda \boldsymbol{b} : \boldsymbol{B}.r\, \textsc{as}\, u : (\boldsymbol{B} \supset A)\, \textsc{in}\, !(u \cdot v_1 \cdot v_2 \ldots \cdot v_m) \end{aligned}$$

An example of the derivation of (1) alluded to above in the case $\Gamma = B$ is:

$$\dfrac{\Delta, v : B; \cdot \vdash [\![\lambda b : B.r]\!]B \supset A \mid !\lambda b : B.r \qquad \dfrac{\Delta, v : B, u : B \supset A; \cdot \vdash A \mid u \cdot v}{\Delta, v : B, u : B \supset A; \cdot \vdash [\![u \cdot v]\!]A \mid !(u \cdot v)}\,\Box\mathsf{I}}{\Delta, v : B; \cdot \vdash [\![(\lambda b : B.r) \cdot v]\!]A \mid \text{X}\textsc{trt}\, !\lambda b : B.r\, \textsc{as}\, u : (B \supset A)\, \textsc{in}\, !(u \cdot v)}\,\Box\mathsf{E}$$

2.2 Normalisation and Evidence Equality

As mentioned above a naïve approach to normalisation is doomed to fail unless our attempt to simplify (hence equate) derivations is *reflected* in the object logic. Indeed, a new judgement must be considered, namely *hypothetical judgements for evidence equality*:

$$\Delta; \Gamma \vdash s \equiv t : A$$

Read: "*s and t are provably equal evidence of the truth of A under the validity assumptions of* Δ *and the truth assumptions of* Γ". This judgement internalises at the object level the equality of derivations induced by the normalisation steps. Note that evidence for provable equality is not considered in hypothetical judgements for evidence equality. Although this could be an interesting route for exploration, in our setting we would then be forced to define a notion of equality on this new kind of evidence, thus leading to an infinite regression.

In addition to defining the meaning of this new judgement by means of new axiom and inference schemes, we must indicate how it affects the meaning of hypothetical judgements with explicit evidence.

$$\frac{\Delta;\Gamma\vdash A\mid s \quad \Delta;\Gamma\vdash s\equiv t:A}{\Delta;\Gamma\vdash A\mid t}\,\mathsf{EqEvid}$$

The upper left judgement of EqEvid is called the minor premise and the one on the right the major premise. Fig. 3 defines the meaning of hypothetical judgement for evidence equality.

Definition 2. $\mathbf{LP}_{nd}$ *is obtained by augmenting the schemes of Fig. 2 with* EqEvid *and the schemes of Fig. 3.*

In the sequel we study hypothetical judgements derivable in $\mathbf{LP}_{nd}$. Note that the structural properties of $\mathbf{LP}^{-}_{nd}$ extend to $\mathbf{LP}_{nd}$.

We now return to normalisation of derivations. Three groups of transformations of derivations are defined: principal contractions, principal expansions and silent permutative contractions. The first two are internalised by the inference schemes defining provable equality of evidence. Permutative conversions need not be internalised since, in contrast to principal contractions, they do not alter the end judgement. They are thus dubbed *silent* permutative conversions. By defining an appropriate notion of cut segment (Def. 7) we show that contraction is weakly normalising: there is a sequence of contractions to normal form. More importantly, we shall see shortly that contraction is in fact strongly normalising. The proof of this is established via weak normalisation.

Definition 3.

1. *Principal Contractions for* $\mathbf{LP}_{nd}$.
   - *Principal contraction for* $\supset$

$$\frac{\dfrac{\Delta;\Gamma,a:A\vdash B\mid s}{\Delta;\Gamma\vdash A\supset B\mid \lambda a:A.s}\supset\mathsf{I} \qquad \Delta;\Gamma\vdash A\mid t}{\Delta;\Gamma\vdash B\mid (\lambda a:A.s)\cdot t}\supset\mathsf{E}$$

   contracts to

$$\frac{\dfrac{\pi}{\Delta;\Gamma\vdash B\mid s^a_t} \qquad \dfrac{\Delta;\Gamma,a:A\vdash B\mid s \quad \Delta;\Gamma\vdash A\mid t}{\Delta;\Gamma\vdash s^a_t\equiv(\lambda a:A.s)\cdot t:B}\mathsf{EqBeta}}{\Delta;\Gamma\vdash B\mid(\lambda a:A.s)\cdot t}\mathsf{EqEvid}$$

   where π *results from the Substitution Principle for Truth with Evidence.*
   - *Principal contraction for* $\Box$.

Axiom Schemes

$$\frac{\Delta; \Gamma \vdash A \mid s}{\Delta; \Gamma \vdash s \equiv s : A}\ \mathsf{EqRefl} \qquad \frac{\Delta; \Gamma, a : A \vdash B \mid s \quad \Delta; \Gamma \vdash A \mid t}{\Delta; \Gamma \vdash s^a_t \equiv (\lambda a : A.s) \cdot t : B}\ \mathsf{EqBeta}$$

$$\frac{\Delta; \cdot \vdash A \mid s \quad \Delta, v : A; \Gamma \vdash C \mid t}{\Delta; \Gamma \vdash t^v_s \equiv \textsc{Xtrt}\, !s\, \textsc{as}\, v : A\, \textsc{in}\, t : C^v_s}\ \mathsf{Eq\Box Beta}$$

$$\frac{\Delta; \Gamma \vdash A \supset B \mid s \quad a \notin \mathsf{fv}(s)}{\Delta; \Gamma \vdash \lambda a : A.(s \cdot a) \equiv s : A \supset B}\ \mathsf{EqEta}$$

$$\frac{\Delta; \Gamma \vdash [\![s]\!]A \mid t \quad u \notin \mathsf{fv}(t)}{\Delta; \Gamma \vdash \textsc{Xtrt}\, t\, \textsc{as}\, u : A\, \textsc{in}\, !u \equiv t : [\![s]\!]A}\ \mathsf{Eq\Box Eta}$$

Inference Schemes For Equivalence

$$\frac{\Delta; \Gamma \vdash s \equiv t : A}{\Delta; \Gamma \vdash t \equiv s : A}\ \mathsf{EqSymm} \qquad \frac{\Delta; \Gamma \vdash s_1 \equiv s_2 : A \quad \Delta; \Gamma \vdash s_2 \equiv s_3 : A}{\Delta; \Gamma \vdash s_1 \equiv s_3 : A}\ \mathsf{EqTrans}$$

Inference Schemes For Congruence

$$\frac{\Delta; \Gamma, a : A \vdash s \equiv t : B}{\Delta; \Gamma \vdash \lambda a : A.s \equiv \lambda a : A.t : A \supset B}\ \mathsf{Eq} \supset \mathsf{I}$$

$$\frac{\Delta; \Gamma \vdash s_1 \equiv s_2 : A \supset B \quad \Delta; \Gamma \vdash t_1 \equiv t_2 : A}{\Delta; \Gamma \vdash s_1 \cdot t_1 \equiv s_2 \cdot t_2 : B}\ \mathsf{Eq} \supset \mathsf{E}$$

$$\frac{\Delta; \cdot \vdash s \equiv t : A}{\Delta; \Gamma \vdash !s \equiv !t : [\![s]\!]A}\ \mathsf{Eq\Box I}_l \qquad \frac{\Delta; \cdot \vdash s \equiv t : A}{\Delta; \Gamma \vdash !s \equiv !t : [\![t]\!]A}\ \mathsf{Eq\Box I}_r$$

$$\frac{\Delta; \Gamma \vdash s_1 \equiv s_2 : [\![r]\!]A \quad \Delta, v : A; \Gamma \vdash t_1 \equiv t_2 : C}{\Delta; \Gamma \vdash \textsc{Xtrt}\, s_1\, \textsc{as}\, v : A\, \textsc{in}\, t_1 \equiv \textsc{Xtrt}\, s_2\, \textsc{as}\, v : A\, \textsc{in}\, t_2 : C^v_r}\ \mathsf{Eq\Box E}$$

Fig. 3. Axiom and inference schemes for evidence equality

$$\dfrac{\dfrac{\Delta; \cdot \vdash A \mid s}{\Delta;\Gamma \vdash [\![s]\!]A \mid\, !s}\ \Box\mathsf{I} \qquad \Delta, v:A;\Gamma\vdash C \mid t}{\Delta;\Gamma \vdash C^v_s \mid \textsc{Xtrt}\, !s \,\textsc{as}\, v:A \,\textsc{in}\, t}\ \Box\mathsf{E}$$

contracts to

$$\dfrac{\dfrac{\pi}{\Delta;\Gamma\vdash C^v_s \mid t^v_s} \qquad \dfrac{\Delta;\cdot\vdash A\mid s \quad \Delta,v:A;\Gamma\vdash C\mid t}{\Delta;\Gamma\vdash t^v_s \equiv \textsc{Xtrt}\,!s\,\textsc{as}\,v:A\,\textsc{in}\,t : C^v_s}\ \mathsf{Eq\Box Beta}}{\Delta;\Gamma\vdash C^v_s \mid \textsc{Xtrt}\,!s\,\textsc{as}\,v:A\,\textsc{in}\,t}\ \mathsf{EqEvid}$$

where π results from the Substitution Principle for Validity with Evidence.

2. *Expansions for* $\mathbf{LP}_{nd}$.
   - *Principal expansion for* $\supset$. *A derivation of the judgement*

$$\Delta;\Gamma\vdash A\supset B\mid s$$

expands to

$$\dfrac{\dfrac{\pi}{\Delta;\Gamma\vdash A\supset B\mid \lambda a:A.(s\cdot a)}\ \supset\mathsf{I} \qquad \dfrac{\Delta;\Gamma\vdash A\supset B\mid s \quad a\notin \mathsf{fv}(s)}{\Delta;\Gamma\vdash \lambda a:A.(s\cdot a)\equiv s : A\supset B}\ \mathsf{EqEta}}{\Delta;\Gamma\vdash A\supset B\mid s}\ \mathsf{EqEvid}$$

where π is

$$\dfrac{\dfrac{\Delta;\Gamma,a:A\vdash A\supset B\mid s \quad \Delta;\Gamma,a:A\vdash A\mid a}{\Delta;\Gamma,a:A\vdash B\mid s\cdot a}\ \supset\mathsf{E}}{\Delta;\Gamma\vdash A\supset B\mid \lambda a:A.(s\cdot a)}\ \supset\mathsf{I}$$

and $\Delta;\Gamma,a:A\vdash A\supset B\mid s$ is obtained from $\Delta;\Gamma\vdash A\supset B\mid s$ by Weakening.

   - *Principal expansion for* $\Box$. *A derivation of the judgement*

$$\Delta;\Gamma\vdash [\![s]\!]A\mid t$$

expands to

$$\dfrac{\dfrac{\Delta;\Gamma\vdash [\![s]\!]A\mid t \qquad \dfrac{\Delta,v:A;\cdot\vdash A\mid v}{\Delta,v:A;\Gamma\vdash [\![v]\!]A\mid\, !v}\ \Box\mathsf{I}}{\Delta;\Gamma\vdash [\![s]\!]A\mid r}\ \Box\mathsf{E} \qquad \dfrac{\Delta;\Gamma\vdash [\![s]\!]A\mid t \quad v\notin\mathsf{fv}(t)}{\Delta;\Gamma\vdash r\equiv t : [\![s]\!]A}\ \mathsf{Eq\Box Eta}}{\Delta;\Gamma\vdash [\![s]\!]A\mid t}\ \mathsf{EqEvid}$$

where r is the proof term $\textsc{Xtrt}\,t\,\textsc{as}\,v:A\,\textsc{in}\,!v$. Note that $([\![v]\!]A)^v_s = [\![s]\!]A$ since, by the labeling condition, v may not occur in A.

3. *Silent Permutative Contractions for* $\mathbf{LP}_{nd}$.
   - *Silent Permutative Contractions for* $\supset$.

$$\dfrac{\dfrac{\Delta;\Gamma\vdash A_1\supset A_2\mid s \quad \Delta;\Gamma\vdash s\equiv t : A_1\supset A_2}{\Delta;\Gamma\vdash A_1\supset A_2\mid t}\ \mathsf{EqEvid} \qquad \Delta;\Gamma\vdash A_1\mid r}{\Delta;\Gamma\vdash A_2\mid t\cdot r}\ \supset\mathsf{E}$$

contracts to

$$\frac{\dfrac{\Delta;\Gamma \vdash A_1 \supset A_2 \mid s \quad \Delta;\Gamma \vdash A_1 \mid r}{\Delta;\Gamma \vdash A_2 \mid s \cdot r}\supset \mathsf{E} \qquad \dfrac{\pi}{\Delta;\Gamma \vdash s \cdot r \equiv t \cdot r : A_2}\mathsf{Eq} \supset \mathsf{E}}{\Delta;\Gamma \vdash A_2 \mid t \cdot r}\mathsf{EqEvid}$$

where π *is*

$$\frac{\Delta;\Gamma \vdash s \equiv t : A_1 \supset A_2 \qquad \dfrac{\Delta;\Gamma \vdash A_1 \mid r}{\Delta;\Gamma \vdash r \equiv r : A_1}\mathsf{EqRefl}}{\Delta;\Gamma \vdash s \cdot r \equiv t \cdot r : A_2}\mathsf{Eq} \supset \mathsf{E}$$

– *Silent Permutative Contractions for* $\Box$.

$$\frac{\dfrac{\Delta;\Gamma \vdash [\![s_1]\!]A \mid s_2 \quad \Delta;\Gamma \vdash s_2 \equiv r : [\![s_1]\!]A}{\Delta;\Gamma \vdash [\![s_1]\!]A \mid r}\mathsf{EqEvid} \qquad \Delta, v : A;\Gamma \vdash C \mid t}{\Delta;\Gamma \vdash C^v_{s_1} \mid \mathrm{X{\scriptsize TRT}}\, r\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t}\Box\mathsf{E}$$

contracts to

$$\frac{\dfrac{\pi_1}{\Delta;\Gamma \vdash C^v_{s_1} \mid q}\Box\mathsf{E} \qquad \dfrac{\pi_2}{\Delta;\Gamma \vdash q \equiv \mathrm{X{\scriptsize TRT}}\, r\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t : C^v_{s_1}}\mathsf{Eq}\Box\mathsf{E}}{\Delta;\Gamma \vdash C^v_{s_1} \mid \mathrm{X{\scriptsize TRT}}\, r\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t}\mathsf{EqEvid}$$

where q *is the proof term* $\mathrm{X{\scriptsize TRT}}\, s_2\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t$ *and* π_1 *is*

$$\frac{\Delta;\Gamma \vdash [\![s_1]\!]A \mid s_2 \quad \Delta, v : A;\Gamma \vdash C \mid t}{\Delta;\Gamma \vdash C^v_{s_1} \mid \mathrm{X{\scriptsize TRT}}\, s_2\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t}\Box\mathsf{E}$$

and π_2 *is*

$$\frac{\Delta;\Gamma \vdash s_2 \equiv r : [\![s_1]\!]A \qquad \dfrac{\Delta, v : A;\Gamma \vdash C \mid t}{\Delta, v : A;\Gamma \vdash t \equiv t : C}\mathsf{EqRefl}}{\Delta;\Gamma \vdash \mathrm{X{\scriptsize TRT}}\, s_2\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t \equiv \mathrm{X{\scriptsize TRT}}\, r\, \mathrm{\scriptsize AS}\, v : A\, \mathrm{\scriptsize IN}\, t : C^v_{s_1}}\mathsf{Eq}\Box\mathsf{E}$$

An *Abstract Reduction System* (ARS) is pair $(A, \to_R)$ where A is a set and $\to_R \subseteq A \times A$. When $a \to_R b$ we say *a reduces in one step* (or simply *reduces*) to b. We usually abbreviate an ARS $(A, \to_R)$ with $\to_R$. We write $\twoheadrightarrow_R$ for the reflexive and transitive closure of $\to_R$ and $a \to_R$ when there exists $b \in A$ such that $a \to_R b$. Finally, we write $|A|$ for the size of A (i.e. number of propositional variables and connectives; the size of a propositional variable is 1).

Definition 4 (Weak and strong normalisation). *An ARS* $\to_R$ *is strongly normalising if there does not exist* $a_1, a_2, \ldots, a_n, \ldots$ *such that*

$$a_1 \to_R a_2 \to_R a_3 \to_R \ldots$$

A $\to_R$*-normal form is an element* $a \in A$ *such that there does not exist* $b \in A$ *such that* $a \to_R b$*. An ARS is weakly normalising if for every* $a \in A$ *there exists a* $\to_R$*-normal form* b *such that* $a \twoheadrightarrow_R b$*.*

Definition 5 (ARS induced by $\mathbf{LP}_{nd}$). *The ARS induced by* $\mathbf{LP}_{nd}$ *is* $(\Pi, \to_{\mathbf{LP}}$ *), where* Π *is the set of all finite* $\mathbf{LP}_{nd}$*-derivations and* $\pi \to_{LP} \pi'$ *if* π' *results from* π *by applying either a principal or a silent permutative contraction.*

Definition 6 (Segment). *A segment of length n in a derivation π of $\mathbf{LP}_{nd}$ is a sequence $\Delta_1; \Gamma_1 \vdash A_1 \mid s_1, \ldots, \Delta_n; \Gamma_n \vdash A_n \mid s_n$ of judgements in π where $A_1, \ldots, A_n$ are occurrences of a "cut" formula A such that:*

1. *$\Delta_i; \Gamma_i \vdash A_i \mid s_i$ (with $i < n$) is the minor premise of an application of* EqEvid *and $\Delta_{i+1}; \Gamma_{i+1} \vdash A_{i+1} \mid s_{i+1}$ is the conclusion of this application.*
2. *$\Delta_n; \Gamma_n \vdash A_n \mid s_n$ is not the minor premise of an application of* EqEvid.
3. *$\Delta_1; \Gamma_1 \vdash A_1 \mid s_1$ is not the conclusion of an application of* EqEvid.

Definition 7 (Cut segment, rank, critical segment).

- *A cut segment is a segment such that $\Delta_n; \Gamma_n \vdash A_n \mid s_n$ is the major premise of an elimination scheme $\supset_e$ or $\Box_e$ and $\Delta_1; \Gamma_1 \vdash A_1 \mid s_1$ is the conclusion of an introduction scheme $\supset_i$ or $\Box_i$, respectively.*
- *The rank of a cut segment is $|A|$. The rank of a derivation π is the maximum of the ranks of the cut segments in π; if there are none, then the rank is zero.*
- *A cut segment is critical in π, sometimes abbreviated π-critical segment, if its rank is that of π.*

We now prove weak normalisation of $(\Pi, \to_{\mathbf{LP}})$. We shall see in Sec. 3 that in fact $(\Pi, \to_{\mathbf{LP}})$ is strongly normalising. This shall be obtained by noting that the contraction rules of Def. 3 define an orthogonal non-erasing second-order rewrite system. As a consequence we may deduce that the ARS induced by $\mathbf{LP}_{nd}$ is strongly normalising from the fact that it is weakly normalising using results from the literature on higher-order rewriting.

Proposition 1. *$(\Pi, \to_{\mathbf{LP}})$ is weakly normalising.*

Proof. Define the size of a derivation π to be the pair (n, m) where:

- n is the rank of a critical cut segment in π and
- m is the sum of the lengths of critical cut segments in π.

Select a redex operating on a critical segment in π that is the *rightmost* and *uppermost* in π. Let π' be the derivation resulting from contracting this redex and let (n', m') be the size of π'. In each case we may verify that $(n, m) > (n', m')$:

- in the case of a principal contraction, two situations may arise (both of which lead to $(n, m) > (n', m')$) depending on whether the last of the cut segments whose rank is that of π is eliminated or not. We illustrate with the principal contraction for $\supset$. Suppose the selected redex is:

$$
\dfrac{\dfrac{\dfrac{\pi_1}{\Delta; \Gamma, a : A \vdash B \mid s}}{\Delta; \Gamma \vdash A \supset B \mid \lambda a : A.s} \supset \mathsf{I} \qquad \dfrac{\pi_2}{\Delta; \Gamma \vdash A \mid t}}{\Delta; \Gamma \vdash B \mid (\lambda a : A.s) \cdot t} \supset \mathsf{E}
$$

The selected critical segment has cut formula $A \supset B$ and length one. Also, there are no π-critical segments in π_2 (for otherwise the selected redex would not operate on a rightmost cut segment) nor in π_1 (for otherwise the selected redex would not operate on an uppermost cut segment). This redex contracts to:

$$\dfrac{\dfrac{\pi_3}{\Delta;\Gamma \vdash B \mid s^a_t} \qquad \dfrac{\dfrac{\pi_1}{\Delta;\Gamma, a : A \vdash B \mid s} \qquad \dfrac{\pi_2}{\Delta;\Gamma \vdash A \mid t}}{\Delta;\Gamma \vdash s^a_t \equiv (\lambda a : A.s)\cdot t : B}\,\mathsf{EqBeta}}{\Delta;\Gamma \vdash B \mid (\lambda a : A.s)\cdot t}\,\mathsf{EqEvid}$$

where π_3 results from the Substitution Principle for Truth with Evidence. Two situations are possible depending on whether there are or are not other critical segments in π (apart from the one which is contracted). In the former case $n' = n$ and $m' = m - 1$ (note that new cut segments may have been created in π_3 however they are all of lower rank since $|A| < |A \supset B|$ and π_2 contains no π-critical segments); in the latter $n' < n$.

- in the case of a silent permutative contraction, we have $(n, m) > (n, m-1)$. For example, suppose the redex selected is

$$\dfrac{\dfrac{\Delta;\Gamma \vdash A_1 \supset A_2 \mid s \qquad \dfrac{\pi_1}{\Delta;\Gamma \vdash s \equiv t : A_1 \supset A_2}}{\Delta;\Gamma \vdash A_1 \supset A_2 \mid t}\,\mathsf{EqEvid} \qquad \dfrac{\pi_2}{\Delta;\Gamma \vdash A_1 \mid r}}{\Delta;\Gamma \vdash A_2 \mid t\cdot r}\supset\mathsf{E}$$

  Note that there are no critical segments in π_1 (for otherwise the selected redex would not operate on an uppermost cut segment) nor are there any in π_2 (for otherwise the selected redex would not operate on a rightmost cut segment). This redex contracts to

$$\dfrac{\dfrac{\Delta;\Gamma \vdash A_1 \supset A_2 \mid s \qquad \dfrac{\pi_2}{\Delta;\Gamma \vdash A_1 \mid r}}{\Delta;\Gamma \vdash A_2 \mid s\cdot r}\supset\mathsf{E} \qquad \dfrac{\pi_3}{\Delta;\Gamma \vdash s\cdot r \equiv t\cdot r : A_2}\,\mathsf{Eq}\supset\mathsf{E}}{\Delta;\Gamma \vdash A_2 \mid t\cdot r}\,\mathsf{EqEvid}$$

  where π_3 is

$$\dfrac{\dfrac{\pi_1}{\Delta;\Gamma \vdash s \equiv t : A_1 \supset A_2} \qquad \dfrac{\dfrac{\pi_2}{\Delta;\Gamma \vdash A_1 \mid r}}{\Delta;\Gamma \vdash r \equiv r : A_1}\,\mathsf{EqRefl}}{\Delta;\Gamma \vdash s\cdot r \equiv t\cdot r : A_2}\,\mathsf{Eq}\supset\mathsf{E}$$

  Notice that the length of the critical segment whose cut formula is $A_1 \supset A_2$ has decreased by one.

At some point a derivation π_{nf} shall be attained which contains no more cut segments. However, silent permutative contractions may still be applicable to π_{nf}. Thus, once such a π_{nf} is achieved, we repeatedly apply silent permutative contractions (which are easily seen to strictly decrease the sum of the lengths of all segments). The resulting derivation shall be in $\rightarrow_{\mathbf{LP}}$-normal form.

3 The Intensional Lambda Calculus

This section introduces the *intensional lambda calculus* ($\lambda^{\mathbf{I}}$) and studies confluence and strong normalisation. We begin by defining the set of *raw* terms of $\lambda^{\mathbf{I}}$:

$$
\begin{array}{lll}
\textit{Proper Terms} & M ::= & x \mid M \cdot M \mid \lambda a : A.M \\
 & \mid & !M \mid \textsc{Xtrt}\, M \,\textsc{as}\, v : A \,\textsc{in}\, M \mid e \blacktriangleright M \\
\textit{Reduction Evidence} & e ::= & \beta([a : A]M, N) \mid \beta_{\Box}([v : A]M, N) \\
 & \mid & \eta(M) \mid \eta_{\Box}(M) \\
 & \mid & \textsc{Refl}(M) \mid \textsc{Sym}(e) \mid e; e \\
 & \mid & \textsc{Abs}([a : A]e) \mid \textsc{App}(e, e) \mid \textsc{BoxL}(e) \mid \textsc{BoxR}(e) \mid \textsc{xtrt}(e, [v : A]e)
\end{array}
$$

A raw term of the form $M \cdot N$ is an *application*, $\lambda a : A.M$ is an *abstraction*, $!M$ is a *bang* term, $\textsc{Xtrt}\, M \,\textsc{as}\, v : A \,\textsc{in}\, N$ is an *extraction* and $e \blacktriangleright M$ is a *registered* term. Reduction evidence $\beta([a : A]M, N)$ is used to register that a principal $\supset$ contraction was applied together with the actual parameters ($\lambda a : A.M$ and N) and $\beta_{\Box}([v : A]M, N)$ is for principal $\Box$ contractions. Similarly for $\eta(M)$ and $\eta_{\Box}(M)$ and η expansions. The remaining reduction evidence terms are for the congruence inference schemes of evidence equality.

Let P range over an enumerable set of type variables. The set of *raw types* is the set of propositions of $\mathbf{LP}_{nd}$. We recall them from Sec. 2.1 (s are ranges over proof terms):

$$A ::= P \mid A \supset A \mid [\![s]\!]A$$

In $\lambda^{\mathbf{I}}$ proper terms are assigned *pointed types* $\langle A, s\rangle$ and reduction evidence is assigned *equality types* $s \equiv t : A$. Since the typing schemes follow the axiom and inference schemes of $\mathbf{LP}_{nd}$, there are two *typing judgements*:

1. $\Delta; \Gamma \vdash M \rhd \langle A, s\rangle$, read: "*Proper term M has pointed type $\langle A, s\rangle$ under type assumptions Δ and Γ*" and
2. $\Delta; \Gamma \vdash e \rhd s \equiv t : A$, read: "*Reduction evidence e has equality type $s \equiv t : A$ under type assumptions Δ and Γ*".

Definition 8. *A proper term M is typable if there exist type assumptions Δ and Γ and a pointed type $\langle A, s\rangle$ such that $\Delta; \Gamma \vdash M \rhd \langle A, s\rangle$ is derivable using the typing schemes presented in Fig. 4. Typability of reduction evidence ($\Delta; \Gamma \vdash e \rhd s \equiv t : A$) is defined in Fig. 5. A $\lambda^{\mathbf{I}}$-term is a raw term that is typable.*

The *contractions* defining normalisation on derivations of $\mathbf{LP}_{nd}$ induce a corresponding *reduction* relation on the $\lambda^{\mathbf{I}}$-terms that encode the derivations.

Definition 9 ($\lambda^{\mathbf{I}}$-reduction). *The $\lambda^{\mathbf{I}}$-reduction relation ($\rightarrow$) is obtained by taking the contextual closure of the reduction axioms:*

Minimal Propositional Logic Fragment

$$\frac{}{\Delta; \Gamma, a : A, \Gamma' \vdash a \rhd \langle A, a\rangle}\ \mathsf{oVar}$$

$$\frac{\Delta; \Gamma, a : A \vdash M \rhd \langle B, s\rangle}{\Delta; \Gamma \vdash \lambda a : A.M \rhd \langle A \supset B, \lambda a : A.s\rangle}\ \supset \mathsf{I} \qquad \frac{\Delta; \Gamma \vdash M \rhd \langle A \supset B, s\rangle \quad \Delta; \Gamma \vdash N \rhd \langle A, t\rangle}{\Delta; \Gamma \vdash M \cdot N \rhd \langle B, s \cdot t\rangle}\ \supset \mathsf{E}$$

Provability Fragment

$$\frac{}{\Delta, v : A, \Delta'; \Gamma \vdash v \rhd \langle A, v\rangle}\ \mathsf{mVar} \qquad \frac{\Delta; \cdot \vdash M \rhd \langle A, s\rangle}{\Delta; \Gamma \vdash !M \rhd \langle [\![s]\!]A, !s\rangle}\ \Box\mathsf{I}$$

$$\frac{\Delta; \Gamma \vdash M \rhd \langle [\![s]\!]A, s'\rangle \quad \Delta, v : A; \Gamma \vdash N \rhd \langle C, t\rangle}{\Delta; \Gamma \vdash \textsc{Xtrt}\, M \textsc{ as}\, v : A \textsc{ in}\, N \rhd \langle C^v_s, \textsc{Xtrt}\, s' \textsc{ as}\, v : A \textsc{ in}\, t\rangle}\ \Box\mathsf{E}$$

$$\frac{\Delta; \Gamma \vdash M \rhd \langle A, s\rangle \quad \Delta; \Gamma \vdash e \rhd s \equiv t : A}{\Delta; \Gamma \vdash e \blacktriangleright M \rhd \langle A, t\rangle}\ \mathsf{EqEvid}$$

Fig. 4. Typing schemes for proper terms

$$\begin{array}{lll}
(\lambda a : A.M) \cdot N & \to_\beta & \beta([a : A]M, N) \blacktriangleright M^a_N \\
\textsc{Xtrt}\, !N \textsc{ as}\, v : A \textsc{ in}\, M & \to_{\beta_\Box} & \beta_\Box([v : A]M, N) \blacktriangleright M^v_N \\
 & & \\
M \rhd A \supset B & \to_\eta & \eta(M) \blacktriangleright \lambda a : A.M \cdot a \\
M \rhd [\![s]\!]A & \to_{\eta_\Box} & \eta_\Box(M) \blacktriangleright \textsc{Xtrt}\, M \textsc{ as}\, v : A \textsc{ in}\, !v \\
 & & \\
(e \blacktriangleright M) \cdot N & \to_{\blacktriangleright L} & \textsc{App}(e, \textsc{Refl}(N)) \blacktriangleright M \cdot N \\
\textsc{Xtrt}\, e \blacktriangleright N \textsc{ as}\, v : A \textsc{ in}\, M & \to_{\blacktriangleright xtr} & \textsc{xtrt}(e, [v : A]\textsc{Refl}(M)) \blacktriangleright \textsc{Xtrt}\, N \textsc{ as}\, v : A \textsc{ in}\, M
\end{array}$$

Note that, just as proof terms are internalised as part of the process of proving a formula in **LP**, so the process of reducing a $\lambda^{\mathbf{I}}$-term internalises evidence of reduction. Indeed, an application of the β reduction rule results in a $\lambda^{\mathbf{I}}$-term that incorporates a witness to the fact that such a reduction step was applied. This reduction evidence provides *intensional* information on *how* the result was computed.

Consider the term from the ordinary typed lambda calculus $I \cdot (I \cdot b)$ (which is also a term in $\lambda^{\mathbf{I}}$) where I abbreviates $\lambda a : A.a$. In the typed lambda calculus it reduces in two different ways to $I \cdot b$ (we underline the contracted redex):

1. $I \cdot (\underline{I \cdot b}) \to I \cdot b$ 2. $\underline{I \cdot (I \cdot b)} \to I \cdot b$

The fact that both these reductions reach the same term is known as a "syntactic coincidence" [HL91] in the rewriting/lambda calculus community. Although the same term is reached they are computed in rather different ways in the sense that unrelated redexes are contracted. Note, however, that in $\lambda^{\mathbf{I}}$ these two derivations now end in different terms:

1. $I \cdot (I \cdot b) \to I \cdot (\beta([a : A]a, b) \blacktriangleright b)$

Axiom Schemes for Evidence Equality

$$\frac{\Delta;\Gamma,a:A\vdash M\rhd\langle B,s\rangle \quad \Delta;\Gamma\vdash N\rhd\langle A,t\rangle}{\Delta;\Gamma\vdash\beta([a:A]M,N)\rhd s^a_t\equiv(\lambda a:A.s)\cdot t:B}\ \mathsf{EqBeta}$$

$$\frac{\Delta;\cdot\vdash N\rhd\langle A,s\rangle \quad \Delta,v:A;\Gamma\vdash M\rhd\langle C,t\rangle}{\Delta;\Gamma\vdash\beta_\Box([v:A]M,N)\rhd t^v_s\equiv\textsc{Xtrt}\,!s\,\textsc{as}\,v:A\,\textsc{in}\,t:C^v_s}\ \mathsf{Eq\Box Beta}$$

$$\frac{\Delta;\Gamma\vdash M\rhd\langle A\supset B,s\rangle \quad a\notin\mathsf{fv}(s)}{\Delta;\Gamma\vdash\eta(M)\rhd\lambda a:A.(s\cdot a)\equiv s:A\supset B}\ \mathsf{EqEta}$$

$$\frac{\Delta;\Gamma\vdash M\rhd\langle[\![s]\!]A,t\rangle \quad u\notin\mathsf{fv}(t)}{\Delta;\Gamma\vdash\eta_\Box(M)\rhd\textsc{Xtrt}\,t\,\textsc{as}\,u:A\,\textsc{in}\,!u\equiv t:[\![s]\!]A}\ \mathsf{Eq\Box Eta}$$

Inference Schemes For Equivalence

$$\frac{\Delta;\Gamma\vdash M\rhd\langle A,s\rangle}{\Delta;\Gamma\vdash\textsc{Refl}(M)\rhd s\equiv s:A}\ \mathsf{EqRefl}$$

$$\frac{\Delta;\Gamma\vdash e\rhd s\equiv t:A}{\Delta;\Gamma\vdash\textsc{Sym}(e)\rhd t\equiv s:A}\ \mathsf{EqSymm} \qquad \frac{\Delta;\Gamma\vdash d\rhd s_1\equiv s_2:A \quad \Delta;\Gamma\vdash e\rhd s_2\equiv s_3:A}{\Delta;\Gamma\vdash d;e\rhd s_1\equiv s_3:A}\ \mathsf{EqTrans}$$

Inference Schemes For Congruence

$$\frac{\Delta;\Gamma,a:A\vdash e\rhd s\equiv t:B}{\Delta;\Gamma\vdash\textsc{Abs}([a:A]e)\rhd\lambda a:A.s\equiv\lambda a:A.t:A\supset B}\ \mathsf{Eq\supset I}$$

$$\frac{\Delta;\Gamma\vdash d\rhd s_1\equiv s_2:A\supset B \quad \Delta;\Gamma\vdash e\rhd t_1\equiv t_2:A}{\Delta;\Gamma\vdash\textsc{App}(d,e)\rhd s_1\cdot t_1\equiv s_2\cdot t_2:B}\ \mathsf{Eq\supset E}$$

$$\frac{\Delta;\cdot\vdash e\rhd s\equiv t:A}{\Delta;\Gamma\vdash\textsc{BoxL}(e)\rhd!s\equiv!t:[\![s]\!]A}\ \mathsf{Eq\Box I}_l \qquad \frac{\Delta;\cdot\vdash e\rhd s\equiv t:A}{\Delta;\Gamma\vdash\textsc{BoxR}(e)\rhd!s\equiv!t:[\![t]\!]A}\ \mathsf{Eq\Box I}_r$$

$$\frac{\Delta;\Gamma\vdash d\rhd s_1\equiv s_2:[\![r]\!]A \quad \Delta,v:A;\Gamma\vdash e\rhd t_1\equiv t_2:C}{\Delta;\Gamma\vdash\textsc{xtrt}(d,[v:A]e)\rhd\textsc{Xtrt}\,s_1\,\textsc{as}\,v:A\,\textsc{in}\,t_1\equiv\textsc{Xtrt}\,s_2\,\textsc{as}\,v:A\,\textsc{in}\,t_2:C^v_r}\ \mathsf{Eq\Box E}$$

Fig. 5. Typing schemes for reduction evidence

2. $I \cdot (I \cdot b) \to \beta([a : A]a, (I \cdot b)) \blacktriangleright I \cdot b$

Since reduction is obtained as a straightforward mapping of contraction of derivations, the following type-soundness result holds.

Lemma 3 (Subject Reduction). *If $M \to_{\lambda^{\mathbf{I}}} N$ and $\Delta; \Gamma \vdash M \rhd \langle A, s \rangle$, then $\Delta; \Gamma \vdash N \rhd \langle A, s \rangle$.*

3.1 Confluence and Strong Normalisation for $\boldsymbol{\lambda}^{\mathbf{I}}$

Higher-order term rewrite systems (HORS) [Klo80,Nip91,TER03] extend first-order term rewrite systems by allowing terms with binders. The λ-calculus is the prototypical example of a HORS. $\lambda^{\mathbf{I}}$ can also be presented as a HORS - we'll present it as an HRS [Nip91]. An HRS is specified by a pair consisting of a *signature* and a set of *rewrite rules* over that signature. A signature is a non-empty set of function symbols, where each function symbol has a unique type. Types are drawn from the set of types of the simply typed lambda calculus ($\lambda^{\to}$). The simply typed lambda calculus is used for representing the objects that are subject to transformation (or rewriting) by means of the rewrite rules. These objects are the terms of $\lambda^{\to}$ that are in $\beta\eta$-long normal form (we shall thus use applicative style notation when writing them). As an example, suppose we wish to model the untyped lambda calculus. For that we define the signature:

$$\mathbf{abs} : (\mathbf{term} \to \mathbf{term}) \to \mathbf{term}$$
$$\mathbf{app} : \mathbf{term} \to \mathbf{term} \to \mathbf{term}$$

where **term** is a base type that represents, intuitively, the set of untyped lambda calculus terms. The objects that are to be rewritten are the $\lambda^{\to}$-terms (in $\beta\eta$-long normal form) formed from these constants and the abstraction and application of $\lambda^{\to}$. For example, the untyped lambda term $(\lambda x.x)\, y$ becomes $\mathbf{app}(\mathbf{abs}(x.x), y)$, where the dot notation is used for the abstraction operation of $\lambda^{\to}$. A rewrite rule is a pair of terms $(f(\boldsymbol{M}), N)$ of the same base type such that all the free variables of N are in $f(\boldsymbol{M})$ and $f(\boldsymbol{M})$ is a pattern (every free variable x occurs in a subterm of the form $x(P_1, \dots P_n)$ with $P_1, \dots, P_n$ η-equivalent to different bound variables). As an example, the β rewrite rule of the untyped lambda calculus is as follows:

$$\mathbf{app}(\mathbf{abs}(x.z(x)), z') \to z(z')$$

We may now state the rewrite relation: a term M *rewrites* to N, written $M \to N$, if there is a rewrite rule (l, r), a substitution σ and a context C such that $M = C[l^{\sigma}]$ and $N = C[r^{\sigma}]$. Note that l^{σ} replaces all free variables with their associated values and then finds the β-normal form of the resulting term.

We are now in condition of presenting the rewrite rules for $\lambda^{\mathbf{I}}$ presented as an HRS, The set of base types is **pterm** (for proper terms) and **redEvid** (for reduction evidence). The signature is given in Fig. 6. The rewrite rules are:

abs: (**pterm**→**pterm**)→**pterm**
app: **pterm**→**pterm**→**pterm**
bang: **pterm**→**pterm**
xtrt: **pterm**→(**pterm**→**pterm**)→**pterm**
evid: **redEvid**→**pterm** →**pterm**
reflE: **pterm**→**redEvid**
appE: **redEvid**→**redEvid**→**redEvid**
xtrtE: **redEvid**→(**pterm**→**redEvid**)→**redEvid**
betaE: (**pterm**→**pterm**)→**pterm**→**redEvid**
betaBoxE: (**pterm**→**pterm**)→**pterm**→**redEvid**

Fig. 6. Signature for $\lambda^{\mathbf{I}}$ as an HRS

$$\begin{array}{lll}
\mathbf{app}(\mathbf{abs}(x.z(x)), y) & \to_{\beta} & \mathbf{evid}(\mathbf{betaE}(x.z(x), y), z(y)) \\
\mathbf{xtrt}(\mathbf{bang}(y), x.z(x)) & \to_{\beta_{\Box}} & \mathbf{evid}(\mathbf{betaBoxE}(x.z(x), y), z(y)) \\
\mathbf{app}(\mathbf{evid}(x, y), z) & \to_{\blacktriangleright L} & \mathbf{evid}(\mathbf{appE}(x, \mathbf{reflE}(z)), \mathbf{app}(y, z)) \\
\mathbf{xtrt}(\mathbf{evid}(w, y), x.z(x)) & \to_{\blacktriangleright xtr} & \mathbf{evid}(\mathbf{xtrtE}(w, x.\mathbf{reflE}(z(x))), \mathbf{xtrt}(y, x.z(x)))
\end{array}$$

The interest in HOR is that general results on combinatorial properties of rewriting can be established. Two such results are of use to us. The first states that orthogonal HORS are confluent. Orthogonal means that rewrite steps are independent: If two redexes in a term may be reduced, the reduction of one of them does not “interfere” with the other one except possibly by duplicating or erasing it.

Proposition 2 ([Nip91]). *Orthogonal HRS are confluent.*

The $\lambda^{\mathbf{I}}$-calculus is easily seen to be an orthogonal HRS[3]. We may thus immediately conclude, from Prop. 2, that it is confluent.

Proposition 3. $\lambda^{\mathbf{I}}$ *is confluent.*

The other interesting property is that of *uniform normalisation*. First we introduce some terminology. A rewrite step $M \to N$ is *perpetual* if whenever M has an infinite reduction, N has one too. A rewrite system is *uniformly normalising* if all its steps are perpetual. An example is the λI-calculus [CR36] which is the standard λ-calculus in which the set of terms is restricted to those M such that $\lambda x.N \subseteq M$ implies $x \in \mathsf{fv}(N)$. The proof of this fact for λI relies on two properties: (1) all reduction steps are *non-erasing* and (2) it is orthogonal. It turns out that this result can be extended to arbitrary higher-order rewrite systems.

Proposition 4 ([KOvO01]). *Non-erasing, orthogonal and fully-extended*[4] *second-order*[5] *HRS are uniformly normalising.*

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[3] It is a left-linear, non-overlapping HRS.

[4] A rewrite system is said to be fully-extended if each of its rewrite rules (l, r) verifies the following: for every occurrence $x(P_1, \ldots, P_n)$ in l of a free variable x, $P_1, \ldots, P_n$ is the list of *all* bound variables above it.

[5] Define the *order* of a type A of the simply typed lambda calculus, written $ord(A)$, to be 1 if the type is a base type and $max(ord(A_1) + 1, A_2)$ if $A = A_1 \to A_2$. The order of rewrite system is the maximum order of the types of the variables that occur in its rewrite rules.

A close look at the HRS presentation of $\lambda^{\mathbf{I}}$ reveals that it is in fact a non-erasing, fully-extended, second-order HRS. Furthermore, we have already mentioned that it is orthogonal. As a consequence, we conclude the following from Prop. 4.

Proposition 5. $\lambda^{\mathbf{I}}$ *is uniformly normalising.*

The interesting thing about uniformly normalisable rewrite systems is that weak normalisation is equivalent to strong normalisation. Therefore, since we have proved that $\lambda^{\mathbf{I}}$ is weakly normalising, we conclude that:

Proposition 6. $\lambda^{\mathbf{I}}$ *is strongly normalising.*

4 Conclusions

A study of the computational interpretation of the Logic of Proofs via the propositions-as-types correspondence requires an appropriate ND presentation. This paper presents one such system, $\mathbf{LP}_{nd}$, resulting from a judgemental analysis [ML83,DP01a] of **LP**. The term assignment yields a typed lambda calculus, called the intensional lambda calculus ($\lambda^{\mathbf{I}}$), that is capable of internalising computation evidence, in much the same way that **LP** is capable of internalising derivability evidence. Computations in $\lambda^{\mathbf{I}}$ yield terms that include information on how this computation is performed.

As mentioned, the fact that $I \cdot (\underline{I \cdot b}) \to I \cdot b$ and $\underline{I \cdot (I \cdot b)} \to I \cdot b$ reduce to the same term in the standard lambda calculus is known as a "syntactic coincidence" [HL91] since these terms are computed in different ways. In $\lambda^{\mathbf{I}}$ the corresponding reductions are no longer cofinal given that intensional information on how the term was computed is part of the result. Further investigation on the relation with equivalence of reductions as defined by Lévy [Lév78,TER03] is left to future work.

Other interesting directions are the formulation of intensional calculi for linear and classical logic given their tight connections with resource conscious computing and control operators and the analysis of the explicit modality and how it relates to staged computation and run-time code generation [DP96,WLPD98].

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