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Cocomplete toposes whose exact completions are toposes

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Abstract

Let $\mathcal{E}$ be a cocomplete topos. We show that if the exact completion of $\mathcal{E}$ is a topos then every indecomposable object in $\mathcal{E}$ is an atom. As a corollary we characterize the locally connected Grothendieck toposes whose exact completions are toposes. This result strengthens both the Lawvere–Schanuel characterization of Boolean presheaf toposes and Hofstra's characterization of the locally connected Grothendieck toposes whose exact completion is a Grothendieck topos.

We also show that for any topological space X, the exact completion of $\mathrm{Sh}(X)$ is a topos if and only if X is discrete. The corollary in this case characterizes the Grothendieck toposes with enough points whose exact completions are toposes.

MSC: 18A35; 18B25

1. Introduction

As explained in Carboni's 1995 paper [2], the forgetful 2-functor $\mathbf{EX} \to \mathbf{LEX}$ from the 2-category of exact categories and exact functors to the 2-category of categories with finite limits and functors preserving these has a left biadjoint $(\_)_{\mathrm{ex}} : \mathbf{LEX} \to \mathbf{EX}$. For each category with finite limits $\mathcal{C}$, the *exact completion* $\mathcal{C}_{\mathrm{ex}}$ of $\mathcal{C}$ has a very simple description. Using this description it is not hard to see that the unit $\mathbf{y} : \mathcal{C} \to \mathcal{C}_{\mathrm{ex}}$ of the adjunction is an embedding and that for each object X of $\mathcal{C}$, the poset of subobjects of $\mathbf{y}X$ can be described as the poset reflection of $\mathcal{C}/X$.

At the end of Section 2 in [2], Carboni attributes to Lawvere the observation that the poset reflection of the topos of graphs $\mathbf{Set}^{\leftleftarrows}$ is not small (see also Section 4 in [6]) and concludes that the exact completion of $\mathbf{Set}^{\leftleftarrows}$ is not a topos. The characterization of the toposes whose exact completions are toposes is stated as an open problem in the last paragraph of p. 131 of [2].

Partial solutions to Carboni's problem were obtained in [8,7,4]. In particular, Hofstra proposes in [4] considering also a variant of the problem: characterizing the Grothendieck toposes whose exact completion is a Grothendieck topos. The existence of small coproducts is preserved by the exact completion construction (see Lemma 2.2 in [2]), but we do not know in general whether the existence of a bound is (see open question 2 in Section 6.3.2 in [4]).

As corollaries of the main results in this paper we will obtain that for a Grothendieck topos $\mathcal{E}$ which is either locally connected or has enough points, $\mathcal{E}_{\mathrm{ex}}$ a topos is equivalent to $\mathcal{E}$ being Boolean; and that in this case $\mathcal{E}_{\mathrm{ex}}$ is Grothendieck.

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Before we state our first main result recall that an object is *indecomposable* if it has no nontrivial coproduct decomposition and that it is an *atom* if it has exactly two subobjects.

Theorem 1.1. *Let $\mathcal{E}$ be a cocomplete topos. If $\mathcal{E}_{\mathrm{ex}}$ is a topos then every indecomposable object of $\mathcal{E}$ is an atom.*

For a cocomplete topos $\mathcal{E}$ we denote the canonical geometric morphism to **Set** by $\Delta \dashv \Gamma : \mathcal{E} \to \mathbf{Set}$. We say that a cocomplete topos $\mathcal{E}$ is *locally connected* if $\Delta : \mathbf{Set} \to \mathcal{E}$ preserves exponentials. By Theorem 15 in [1] this is equivalent to the assertion that every object of $\mathcal{E}$ is a sum of indecomposables. The topos $\mathcal{E}$ is said to be *atomic* if Δ is logical. It follows that a locally connected cocomplete topos is atomic if and only if it is Boolean if and only if every object is a sum of atoms. From Theorem 1.1 we can immediately conclude the following.

Corollary 1.2. *If $\mathcal{E}$ is a cocomplete, locally connected topos such that $\mathcal{E}_{\mathrm{ex}}$ is a topos then $\mathcal{E}$ is Boolean.*

Restricting attention to bounded toposes over **Set** we obtain a characterization of the locally connected Grothendieck toposes whose exact completions are toposes. In order to state it let $\mathbf{Indec}(\mathcal{E})$ be the full subcategory of $\mathcal{E}$ determined by its indecomposable objects.

Corollary 1.3. *If $\mathcal{E}$ is a locally connected Grothendieck topos then the following are equivalent:*

1. *$\mathcal{E}$ is Boolean,*
2. *$\mathbf{Indec}(\mathcal{E})$ is essentially small,*
3. *$\mathcal{E}_{\mathrm{ex}}$ is canonically equivalent to the topos of presheaves on $\mathbf{Indec}(\mathcal{E})$,*
4. *$\mathcal{E}_{\mathrm{ex}}$ is a Grothendieck topos,*
5. *$\mathcal{E}_{\mathrm{ex}}$ is a topos.*

An important special case of Corollary 1.3 is that given by presheaf toposes; as usual we denote the topos of presheaves on $\mathcal{C}$ by $\widehat{\mathcal{C}}$.

Corollary 1.4. *For any essentially small category $\mathcal{C}$, the following are equivalent:*

1. *$\mathcal{C}$ is a groupoid,*
2. *$\mathbf{Indec}(\widehat{\mathcal{C}})$ is essentially small,*
3. *$(\widehat{\mathcal{C}})_{\mathrm{ex}}$ is a topos.*

This is Theorem 6.1.1 in [7] but, as explained there, the equivalence between items 1 and 2 is due to Lawvere and Schanuel. It was Lawvere who suggested that their characterization could be of interest in connection with the problem of characterizing the presheaf toposes whose exact completions are toposes. As far as I know their result is still unpublished but the proof Lawvere communicated to me is sketched in [7]. (I was recently informed that Lawvere and Schanuel were aware of the equivalence between 1 and 2 in Corollary 1.3.) The proof of the main results here uses a different method.

The possibility of generalizing Theorem 6.1.1 in [7] to Corollary 1.3 above was suggested by Theorem 6.3.6 in Hofstra's thesis [4] which essentially says that for a locally connected Grothendieck topos $\mathcal{E}$, $\mathcal{E}$ is atomic if and only if $\mathcal{E}_{\mathrm{ex}}$ is a *Grothendieck* topos.

All toposes whose exact completions are toposes we know of are Boolean (they satisfy the external axiom of choice or are atomic). In the face of Corollary 1.3 one may be tempted to build non-Boolean examples using non-locally connected spaces. Our second main result can be stated as follows.

Theorem 1.5. *Let X be a sober topological space. If $\mathrm{Sh}(X)_{\mathrm{ex}}$ is a topos then X is discrete.*

It is then not difficult to prove the following.

Corollary 1.6. *Let $\mathcal{E}$ be a cocomplete topos. If $\mathcal{E}_{\mathrm{ex}}$ is a topos then the spatial part of $\mathrm{Sub}_{\mathcal{E}}1$ is discrete. (Here $\mathrm{Sub}_{\mathcal{E}}1$ is the locale of subobjects of the terminal object in $\mathcal{E}$.)*

In particular, we obtain for Grothendieck toposes with enough points a result analogous to Corollary 1.3.

Corollary 1.7. *If $\mathcal{E}$ is a Grothendieck topos with enough points then the following are equivalent:*

1. *$\mathcal{E}$ is Boolean,*
2. *$\mathcal{E}$ is locally connected and* **Indec**$(\mathcal{E})$ *is essentially small,*
3. *$\mathcal{E}_{\text{ex}}$ is canonically equivalent to the topos of presheaves on* **Indec**$(\mathcal{E})$*,*
4. *$\mathcal{E}_{\text{ex}}$ is a Grothendieck topos,*
5. *$\mathcal{E}_{\text{ex}}$ is a topos.*

Corollaries 1.3 and 1.7 give pretty general partial answers to Carboni's problem. They also show that under the assumption of enough points or local connectedness Hofstra's problem is equivalent to Carboni's. At the same time, they generalize the Lawvere–Schanuel characterization of presheaf toposes.

On the other hand, it is clear that we still do not fully understand toposes whose exact completions are toposes. In particular, we do not understand their relation to Boolean toposes. Further research will be needed but in the meantime we present (in Section 2) an example of a Boolean topos whose exact completion is not a topos.

The outline of the paper is as follows. In Section 2 we briefly explain the proofs of Corollaries 1.3 and 1.7 and how they depend on the other results stated here. We then review the characterization of the categories with finite limits whose exact completions are toposes given in [8] and describe the main idea in the proofs of Theorems 1.1 and 1.5 above. It will then be clear how Corollary 1.6 follows from the latter theorem. By the end of this section then only Theorems 1.1 and 1.5 will remain to be proved. The essential technical tool for proving these is introduced in Section 3. Section 4 highlights some particular aspects of the Sierpinski topos needed in the main proofs and then Theorems 1.1 and 1.5 are proved in Sections 5 and 6 respectively.

2. A sketch of the proofs

In this section we first discuss the proofs of Corollaries 1.3 and 1.7, relying on Theorems 1.1 and 1.5 (through Corollary 1.6 in the latter case). Then we recall the characterization of the categories with finite limits whose exact completions are toposes and some related results given in [8,7]. These will play a fundamental role in the main proofs of the paper. In particular, we show below Corollary 2.6 how Corollary 1.6 follows from Theorem 1.5.

Finally, we present a Boolean topos whose exact completion is not a topos.

For convenience, we re-state Corollaries 1.3 and 1.7 below.

Corollary 2.1 (*Corollaries* 1.3 *and* 1.7). *Let $\mathcal{E}$ be a Grothendieck topos. If $\mathcal{E}$ is locally connected or it has enough points then the following are equivalent:*

1. *$\mathcal{E}$ is atomic,*
2. *$\mathcal{E}$ is locally connected and* **Indec**$(\mathcal{E})$ *is essentially small,*
3. *$\mathcal{E}_{\text{ex}}$ is canonically equivalent to the topos of presheaves on* **Indec**$(\mathcal{E})$*,*
4. *$\mathcal{E}_{\text{ex}}$ is a Grothendieck topos,*
5. *$\mathcal{E}_{\text{ex}}$ is a topos.*

Proof. If the canonical $\gamma : \mathcal{E} \to \mathbf{Set}$ is atomic then it is trivially locally connected. It is also well known that in an atomic Grothendieck topos there are, up to isomorphism, only a set of indecomposable objects (see the paragraph before Theorem C3.5.8 in [5]). So it is well known that the first item implies the second.

To prove that the second item implies the third let $\mathcal{C}$ be the category **Indec**$(\mathcal{E})$ of indecomposable objects of $\mathcal{E}$. The second item says that $\mathcal{C}$ is essentially small and that every object of $\mathcal{E}$ is isomorphic to a small coproduct of objects in $\mathcal{C}$. By the characterization of coproduct completions (see Lemma 42 in [3]), $\mathcal{E}$ is equivalent to the coproduct completion $\mathcal{C}_+$ of $\mathcal{C}$. Then, by the Corollary in p. 130 of [2], $\mathcal{E}_{\text{ex}} \cong (\mathcal{C}_+)_{\text{ex}} \cong \widehat{\mathcal{C}}$ so $\mathcal{E}_{\text{ex}}$ is a presheaf topos.

The third item trivially implies the fourth and trivially again the fourth implies the fifth. So we are left to prove that if $\mathcal{E}_{\text{ex}}$ is a topos then $\mathcal{E}$ is Boolean. Here is where the proof splits.

If $\mathcal{E}$ is locally connected then Corollary 1.2 implies that $\mathcal{E}$ is atomic.

To finish the case of $\mathcal{E}$ with enough points we use Lemma C3.5.5 in [5] which states that a geometric morphism $f : \mathcal{F} \to \mathcal{F}'$ is atomic if and only if for every object B of $\mathcal{F}$, the composite $\mathcal{F}/B \to \mathcal{F} \to \mathcal{F}'$ can be factored as an hyperconnected morphism followed by a local homeo.

So let $\gamma : \mathcal{E} \to \mathbf{Set}$ be a Grothendieck topos such that $\mathcal{E}$ has enough points and such that $\mathcal{E}_{\text{ex}}$ is a topos. Let B be an object of $\mathcal{E}$ and consider the composite $\mathcal{E}/B \to \mathcal{E} \to \mathbf{Set}$. Take its hyperconnected–localic factorization $\mathcal{E}/B \to \mathrm{Sh}(X) \to \mathbf{Set}$. As $\mathcal{E}/B \to \mathrm{Sh}(X)$ is hyperconnected, $\mathrm{Sub}_{\mathcal{E}/B}(1) \cong \mathrm{Sub}_{\mathrm{Sh}(X)}(1) \cong X$. As $\mathcal{E}_{\text{ex}}$ is a topos,

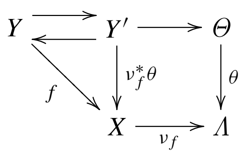
$(\mathcal{E}/B)_{\text{ex}} \cong \mathcal{E}_{\text{ex}}/B$ is a topos. So Corollary 1.6 implies that the spatial part of X is discrete. But $\mathcal{E}$ has enough points, so $\mathcal{E}/B$ has enough points. And as $\mathcal{E}/B \to \text{Sh}(X)$ is surjective, $\text{Sh}(X)$ also has enough points. This means that X is spatial, so X is discrete. That is, $\text{Sh}(X) \to \mathbf{Set}$ is a local homeo. So we can then conclude that $\gamma : \mathcal{E} \to \mathbf{Set}$ is atomic. □

The reader may have noticed that the first item in the result above is stated in terms of atomicity instead of Booleanness. For Grothendieck toposes that are locally connected or that have enough points, Booleanness is equivalent to atomicity (see C3.5.2 in [5]). So there is no problem. On the other hand, consider the case of toposes of sheaves over complete atomless Boolean algebras. These toposes have no points but as they satisfy the external axiom of choice, their exact completions are toposes trivially. So at this time we feel that the statements in terms of Booleanness reflect better our present knowledge of the subject.

2.1. Exact completions and toposes

We have already mentioned that the main results in the present paper rely on the characterization of the categories with finite limits whose exact completions are toposes given in [7,8], so we give a brief recap.

Definition 2.2. A *generic proof* is a map $\theta : \Theta \to \Lambda$ such that for every map $f : Y \to X$ there exists a $\nu_f : X \to \Lambda$ such that f factors through $\nu_f^*\theta$ and $\nu_f^*\theta$ factors through f:



As an immediately corollary of Theorem 1.2 in [8] one obtains the following.

Lemma 2.3. *For any topos $\mathcal{E}$, $\mathcal{E}_{\text{ex}}$ is a topos if and only if $\mathcal{E}$ has generic proof.*

Although it is clearly far from definitive, Lemma 2.3 has proved quite helpful. For example, in 1999 the available information concerning toposes whose exact completions are toposes was that:

1. the exact completion of a topos in which every epi splits is a topos (trivially, since in this case the embedding $\mathcal{E} \to \mathcal{E}_{\text{ex}}$ is an equivalence; the subobject classifier works as a generic proof),
2. the exact completion of atomic Grothendieck toposes are toposes (in fact, presheaf toposes, as in the proof of Corollary 2.1),
3. the exact completion of the topos of irreflexive graphs $\mathbf{Set}^{\leftleftarrows}$ is not a topos.

When Lemma 2.3 was announced in CT99, Carboni suggested that it be tested by applying it to answer the question: Is the exact completion of the Sierpinski topos a topos? (The argument used in the case of irreflexive graphs does not work because the poset reflection of each slice of the Sierpinski topos is small; see Lemma 5.6 in [8].) The answer to this question will be of key relevance in the present paper so let us state it below.

Lemma 2.4. *The exact completion of the Sierpinski topos is not a topos.*

Proof. See Proposition 5.7 in [8] where it is proved that $\mathbf{Set}^{\to}$ does not have a generic proof. □

The key idea to be used in the proofs of the main results is considering certain geometric morphisms $f : \mathcal{E} \to \mathcal{F}$ (to be called *helpful*) which satisfy that if there is a generic proof in $\mathcal{E}$ then there is a generic proof in $\mathcal{F}$. Indeed, the more technical parts of the proof will show that if $\mathcal{E}$ is not Boolean then there is a helpful geometric morphism $f : \mathcal{E} \to \mathbf{Set}^{\to}$, and then, by Lemma 2.4, $\mathcal{E}$ will not have a generic proof.

Lemma 2.5. *Let $Q : \mathcal{D} \to \mathcal{C}$ be a pullback preserving functor with a section $S : \mathcal{C} \to \mathcal{D}$. If $\mathcal{D}$ has a generic proof then so does $\mathcal{C}$.*

In particular, localizations and coreflective subcategories inherit generic proofs. Let us state two relevant instances in terms of geometric morphisms and exact completions.

Corollary 2.6. *Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism. If f is an inclusion and $\mathcal{F}_{\mathrm{ex}}$ is a topos then so is $\mathcal{E}_{\mathrm{ex}}$. On the other hand, if f is connected and $\mathcal{E}_{\mathrm{ex}}$ is a topos then so is $\mathcal{F}_{\mathrm{ex}}$.*

Using this result we can explain how to conclude Corollary 1.6 from Theorem 1.5.

Proof of Corollary 1.6. Let $\gamma : \mathcal{E} \to \mathbf{Set}$ be a cocomplete topos over sets such that $\mathcal{E}_{\mathrm{ex}}$ is a topos. Take the hyperconnected–localic factorization $\mathcal{E} \to \mathrm{Sh}(X) \to \mathbf{Set}$ of γ and let X_s be spatial part of X (inducing a regular mono $X_s \to X$ in the category of locales). As $\mathcal{E} \to \mathrm{Sh}(X)$ is connected and $\mathrm{Sh}(X_s) \to \mathrm{Sh}(X)$ is an inclusion, Corollary 2.6 implies first that $\mathrm{Sh}(X)_{\mathrm{ex}}$ is a topos and then that $\mathrm{Sh}(X_s)_{\mathrm{ex}}$ also is. By Theorem 1.5, X_s is discrete. □

Corollary 2.6 also allows us to restrict our proof of Theorem 1.5 to T_1 spaces.

Lemma 2.7. *Let X be a sober topological space. If $\mathrm{Sh}(X)_{\mathrm{ex}}$ is a topos then X is T_1.*

Proof. If X is not T_1 then the Sierpinski space embeds into X. Taking categories of sheaves we obtain an inclusion of the Sierpinski topos into $\mathrm{Sh}(X)$. Corollary 2.6 and Lemma 2.4 imply that $\mathrm{Sh}(X)_{\mathrm{ex}}$ cannot be a topos. □

The characterization given in Lemma 2.3 shows that the condition of $\mathcal{E}_{\mathrm{ex}}$ being a topos is a weakening of the axiom of choice in $\mathcal{E}$. (Indeed, if we define a *proof classifier* in the same way as a generic proof but where the ν_f is required to be unique, then it is easy to show that a topos has a proof classifier if and only if every epi splits. See Section 5.4 in [7].) From this perspective, Corollaries 1.2 and 1.6 may be seen as analogous to Diaconescu's theorem saying that the internal axiom of choice (IAC) implies that the underlying topos is Boolean.

Boolean locally connected Grothendieck toposes are atomic. So for such a topos $\mathcal{E}$, (IAC) implies that $\mathcal{E}_{\mathrm{ex}}$ is a topos. But the converse is not true as witnessed by the toposes of continuous group actions where (IAC) does not hold (see D4.5.2(c) in [5]).

We end this section with another application of Lemma 2.3. We prove that the exact completion of the Boolean topos of uniform $\mathbb{Z}$-sets is not a topos. Let G be the profinite completion of the additive group $\mathbb{Z}$ of integers. Example A2.1.7 in [5] explains that the continuous G-sets can be identified with the $\mathbb{Z}$-sets in which every orbit is finite. Such a $\mathbb{Z}$-set is uniformly continuous if and only if there is a finite bound for the sizes of its orbits. Let $\mathcal{E}$ be the Boolean topos of uniformly continuous $\mathbb{Z}$-sets. (It seems important to remark that this topos is not bounded.) We now prove that $\mathcal{E}$ does not have a generic proof. For each $n \geq 0$ let T_n be the $\mathbb{Z}$-set given by the orbit of size n. Now suppose that $\mathcal{E}$ does have a generic proof $\theta : \Theta \to \Lambda$. Then there exists a map $t_n : 1 \to \Lambda$ such that the diagram

$$\begin{array}{ccccc} T_n & \rightleftarrows & P_n & \longrightarrow & \Theta \\ & \searrow^{!} & \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle \theta} \\ & & 1 & \xrightarrow[t_n]{} & \Lambda \end{array}$$

commutes and the square is a pullback. It follows that P_n must have a component of size at least n. But the horizontal maps in the square are monos, so Θ cannot be uniform. Absurd.

3. Helpful adjunctions

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with finite limits and let $L \dashv R : \mathcal{C} \to \mathcal{D}$ be an adjunction. Moreover, let $h : X \to Y$ be a map in $\mathcal{D}$ and consider the pullback below:

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & RLX \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle RLh} \\ Y & \xrightarrow[\eta_Y]{} & RLY \end{array}$$

where η is the unit of $L \dashv R$. We say that the adjunction *helps* the map h if π_0 factors through h; in other words, if h and π_0 induce the same object in the poset reflection of $\mathcal{C}/Y$ (because h always factors through π_0).

We say that an adjunction $L \dashv R : \mathcal{C} \to \mathcal{D}$ is *(epi-)helpful* if it helps all (epi-)maps in $\mathcal{D}$. Almost immediately from Definition 3.1 one obtains the following.

Lemma 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with finite limits and let $L \dashv R : \mathcal{C} \to \mathcal{D}$ be a helpful adjunction. If $\mathcal{C}$ has a generic proof then so does $\mathcal{D}$.*

Proof. In fact, we show that R maps the generic proof $\theta : \Theta \to \Lambda$ in $\mathcal{C}$ to a generic proof in $\mathcal{D}$. Let $h : X \to Y$ in $\mathcal{D}$ so that there is a map $\sigma : LY \to \Lambda$ such that the pullback $p : P \to LY$ of θ along σ factors through Lh. Then Rp is the pullback of $R\theta$ along $R\sigma$ and Rp factors through RLh. Helpfulness implies that the pullback of $R\theta$ along $(R\sigma)\eta_Y$ factors through h. □

Coreflections are trivial examples of helpful adjunctions because in this case the units are isomorphisms. More interesting examples are obtained as follows. Let A be exponentiable in $\mathcal{C}$. Then the adjunction $A^* \dashv \Pi_A : \mathcal{C}/A \to \mathcal{C}$ is helpful if and only if A has a point. We will not need this result so we will not prove it.

We say that a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ helps a map if the adjunction $f^* \dashv f_*$ does. For example, recall (Lemma A4.2.6 in [5]) that f is a surjection if and only if for every monic m, the naturality square $\eta m = (f_* f^* m)\eta$ is a pullback (where η is the unit of $f^* \dashv f_*$). In other words, a geometric morphism is a surjection if and only if it helps monos.

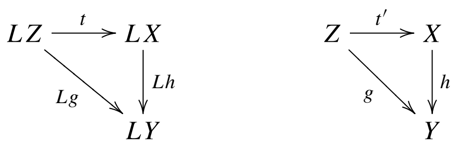
Lemma 3.3. *Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism. Then f is helpful if and only if it is epi-helpful and a surjection.*

Proof. We have already mentioned that mono-helpful geometric morphisms are exactly the surjections. The result follows because in toposes every map factors as an epi followed by a mono. □

It seems important to remark that in order to transfer generic proofs from one topos to another it is enough to help epis. In order to see this define a *generic epi* to be an epi map θ such that satisfies the property defining generic proofs as in Definition 2.2 but restricted to the cases when f is an epimorphism. We claim that a topos has a generic proof if and only if it has a generic epi. To prove this notice that if $\theta : \Theta \to \Lambda$ is a generic proof then the epi part of its epi–mono factorization is a generic epi. Conversely, if $\theta : \Theta \to \Lambda$ is a generic epi then postcomposing with the insertion $\Lambda \to \Lambda_\perp$ into the partial map classifier provides a generic proof. Finally, an argument analogous to that used to prove Lemma 3.2 shows that if $f : \mathcal{E} \to \mathcal{F}$ is epi-helpful and $\mathcal{E}$ has a generic epi then so does $\mathcal{F}$. In spite of this, our proofs will sometimes need a certain geometric morphism to be not only epi-helpful but also surjective.

We have already said that helpful geometric morphisms $\mathcal{E} \to \mathbf{Set}^{\to}$ will play an important role. In the next section we review some particular features of the Sierpinski topos that we will need in our proofs.

Remark 3.4. During the refereeing process I noticed a different formulation of helpful adjunctions. An adjunction $L \dashv R : \mathcal{C} \to \mathcal{D}$ between categories with finite limits is helpful if and only if for every diagram as on the left below



there exists one as on the right above. This different formulation has not produced new results but it seemed worth stating. It also seems worth saying that one of the referees suggested that it may be useful to provide a fibered-theoretical formulation of helpful adjunctions. Unfortunately, I cannot see at present how to do this in such a way that it would give substantial insight.

4. The Sierpinski topos

The Sierpinski topos is the classifying topos for subterminal objects (see Remark B3.2.11). It can be explicitly described as the topos of sheaves on the Sierpinski space or as the topos of presheaves on the poset $\mathbf{2} = \{\bot < \top\}$. Lemma 2.4 says that it does not have a generic proof. In this section we review some simple properties of this topos that will help us prove that certain geometric morphisms with $\widehat{\mathbf{2}}$ as codomain are helpful.

We will write $\bot$ and $\top$ for the induced representables in $\widehat{\mathbf{2}}$. Notice that $\top$, as an object in $\widehat{\mathbf{2}}$, is the terminal 1. Because $\widehat{\mathbf{2}}$ is a presheaf topos, it is locally connected. Let us describe the indecomposable objects of $\widehat{\mathbf{2}}$. Think of the objects as functions $a : X \to Y$. It is not difficult to show that such an object is indecomposable if and only if Y is a singleton. In other words, an object A of $\widehat{\mathbf{2}}$ is indecomposable if and only if $A\bot = 1$. It is very useful to picture the $\bot$-figures of an object as dots and the $\top$-figures as loops around the dots.

Lemma 4.1. *The following hold in* $\widehat{\mathbf{2}}$*:*

1. *every epi between indecomposable objects splits,*
2. *for every* X, $X \times \bot = \sum_{x \in X\bot} \bot$.

We now need to characterize surjections onto the Sierpinski topos. The results below are probably folklore so we only sketch the proofs. Let $\mathcal{E}$ be a cocomplete topos and let $f : \mathcal{E} \to \widehat{\mathbf{2}}$ be a geometric morphism. Of course, $f^*\top = f^*1 = 1$ and $f^*\bot$ is the classified subobject of 1 in $\mathcal{E}$.

For any set I, let C_I be the indecomposable object whose representation as a function is $! : I \to 1$. The object C_I should be thought of as one dot with I loops around it. Notice that $C_0 = \bot$ and $C_1 = \top$.

Lemma 4.2. *Let* $\mathcal{E}$ *be a topos and let* $f : \mathcal{E} \to \widehat{\mathbf{2}}$ *be a geometric morphism. Then the following hold:*

1. $f_*(f^*C_1) = C_1$,
2. *for any* I, $f_*(f^*C_I)$ *is indecomposable,*
3. $f_*(f^*C_0) = C_0$ *if and only if* $f^*\bot \to 1$ *is strict,*
4. $f_*(f^*0) = 0$ *if and only if* $0 \to f^*\bot$ *is strict.*

Proof. The first item is trivial because $C_1 = \top = 1$. To prove the second item we must show that $(f_*(f^*C_I))\bot \cong \mathcal{E}(f^*\bot, f^*C_I) = 1$. By Lemma 4.1 $C_I \times \bot = \bot$, so $f^*C_I \times f^*\bot = f^*\bot$. Notice also that as $f^*\bot$ is subterminal, $\mathcal{E}(f^*\bot, f^*\bot)$ is a singleton. Then it is easy to calculate: $\mathcal{E}(f^*\bot, f^*C_I) \cong \mathcal{E}(f^*\bot, f^*C_I \times f^*\bot) \cong \mathcal{E}(f^*\bot, f^*\bot) \cong 1$. To prove the third item, notice that $(f_*(f^*C_0))\top = \mathcal{E}(1, f^*\bot)$. This set is empty if and only if $f^*\bot \to 1$ is strict. Finally notice that $(f_*0)\bot = \widehat{\mathbf{2}}(\bot, f_*0) = \mathcal{E}(f^*\bot, 0)$ is empty if and only if $0 \to f^*\bot$ is strict. $\square$

The characterization of surjections onto the Sierpinski topos can now be stated as follows.

Proposition 4.3. *For a geometric morphism* $f : \mathcal{E} \to \widehat{\mathbf{2}}$ *the following are equivalent:*

1. f *is a surjection,*
2. *both* $0 \rightarrowtail f^*\bot$ *and* $f^*\bot \rightarrowtail 1$ *are strict,*
3. *the functor* $f_*f^* : \widehat{\mathbf{2}} \to \widehat{\mathbf{2}}$ *preserves 0 and* $\bot$.

Proof. Lemma A1.2.4 in [5] implies that the inverse image functor of a geometric morphism is faithful if and only if it preserves strictness of subobjects. Together with the fact that in a presheaf topos every object is a quotient of a small coproduct of representables we conclude that a geometric morphism $f : \mathcal{E} \to \widehat{\mathcal{C}}$ is a surjection if and only if for every representable R and monic $m : U \to R$ in $\widehat{\mathcal{C}}$, f^*m iso implies m iso. Finally, notice that the second item is essentially saying that f^* preserves strictness of subobjects of representables. So the first two items are equivalent. (The equivalence between the second and third items follows from Lemma 4.2.) $\square$

5. Indecomposable objects are atoms

In this section we prove Theorem 1.1. First we show that for toposes $\mathcal{F}$ with indecomposable terminal object all surjective geometric morphisms $\mathcal{F} \to \widehat{\mathbf{2}}$ are helpful (Proposition 5.4). Then we explain how this implies Theorem 1.1.

We say that a map $g : X \to Y$ is *simple* if there exists a collection $\{m_j : X_j \to Y\}_{j \in J}$ of monic maps and an iso $X \cong \sum_j X_j$ making the induced $m : \sum_j X_j \to Y$ iso to g over Y.

Lemma 5.1. *For every map* $X \to Y$ *in* $\widehat{\mathbf{2}}$ *there exists a simple one which induces the same object in the poset reflection of* $\widehat{\mathbf{2}}/Y$.

Proof. Actually the argument works for any locally connected topos in which every epi between indecomposable object splits. As $\widehat{\mathbf{2}}$ is locally connected, X is iso to a coproduct $\sum_{j \in J} X_j$ with X_j indecomposable. Denote $f in_j$ by $f_j : X_j \to Y$ and let $m_j e_j = f_j$ be the epi–mono factorization of f_j with $e_j : X_j \to D_j$. By Lemma 4.1, e_j splits so f_j induces the same proof as m_j. Altogether the map $\sum_{j \in J} e_j : X = \sum_{j \in J} X_j \to \sum_{j \in J} D_j$ splits, and so f induces the same proof as the morphism $\sum_{j \in J} D_j \to Y$ determined by the family of m_j's. $\square$

We could prove Theorem 1.1 directly but we believe that the proof can be more easily understood if we split some of its key parts. In order to do this let us introduce the following technical definition.

Definition 5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with finite limits and let $L \dashv R : \mathcal{C} \to \mathcal{D}$ be an adjunction. We say that the adjunction $L \dashv R$ is *willing to help* a map $a : A \to Y$ if there are maps $\pi : S \to Y$ and $s : S \to RLA$ with S projective and such that the following diagram:

$$\begin{array}{ccc} S + A & \xrightarrow{[s,\eta]} & RLA \\ {\scriptstyle [\pi,a]}\downarrow & & \downarrow{\scriptstyle RLa} \\ Y & \xrightarrow[\eta]{} & RLY \end{array}$$

is a pullback. (As usual, η is the unit of the adjunction.)

Clearly, if an adjunction is willing to help a map e and e is epi then the adjunction actually helps e. We say that a geometric morphism f is willing to help a map if the adjunction $f^* \dashv f_*$ is.

Lemma 5.3. *Let $\mathcal{F}$ be a topos with indecomposable terminal object and let $f : \mathcal{F} \to \widehat{\mathbf{2}}$ be a surjective geometric morphism. Then f is willing to help simple maps.*

Proof. Let $a : A \to Y$ be simple in $\widehat{\mathbf{2}}$. We need to prove that the following square:

$$\begin{array}{ccc} S + A & \xrightarrow{s+\eta} & f_* f^* A \\ {\scriptstyle [\pi,a]}\downarrow & & \downarrow{\scriptstyle f_* f^* a} \\ Y & \xrightarrow[\eta]{} & f_* f^* Y \end{array}$$

is a pullback for some $\pi : S \to Y$ with S projective.

As a is simple we can assume that $A = \sum_{i \in I} A_i$ and that a is induced by a collection $\{a_i : A_i \to Y\}_{i \in I}$ of monics. To prove the result let P be the pullback of $f_* f^* a$ along η. In presheaf toposes limits are calculated pointwise so $P\top$ is the set of pairs (p, α) with $p : \top \to Y$ and $\alpha : f^*\top = 1 \to f^* \sum_{i \in I} A_i \cong \sum_{i \in I} f^* A_i$ such that the following diagram commutes:

$$\begin{array}{ccc} f^*1 & \xrightarrow{\alpha} & \sum_{i \in I} f^* A_i \\ & {\scriptstyle f^*p}\searrow & \downarrow{\scriptstyle [\ldots, f^* a_i, \ldots]} \\ & & f^* Y \end{array}$$

and $P\bot$ is analogous. Because $1 = f^*\top$ is indecomposable, α factors through $f^* A_k$ for some k, so $f^* p$ factors through $f^* a_k$. As f is surjective, p factors through a_k. That is, there exists a unique $l : 1 \to A_k$ such that $p = a_k l$. In other words, $p \le a_k$ over Y. So $\alpha = f^* l$. In other words, $P\top = (\sum_{i \in I} A_i)\top$.

On the other hand, $P\bot$ consists of $(\sum_{i \in I} A_i)\bot$ plus the $f^*\bot \to \sum_{i \in I} f^* A_i$ that are not in the image of f^*. So $P \cong S + A$ where $S = \sum_{j \in J} \bot$ for some J, a sum of projectives. □

We can now prove the connected case of Theorem 1.1.

Proposition 5.4. *Let $\mathcal{F}$ be a topos with indecomposable terminal object. Then every surjective geometric morphism $f : \mathcal{F} \to \widehat{\mathbf{2}}$ is helpful.*

Proof. By Lemma 3.3, it is enough to prove that f is epi-helpful. Lemma 5.1 implies that it is enough to prove that f helps (epi-)simple maps. For this it is enough to prove that f is willing to help simple maps. But this is Lemma 5.3. □

We are now ready to prove Theorem 1.1, that is: if $\mathcal{E}$ is a cocomplete topos such that $\mathcal{E}_{\mathrm{ex}}$ is a topos then every indecomposable object of $\mathcal{E}$ is an atom.

Proof of Theorem 1.1. Let X be an indecomposable object in $\mathcal{E}$. Lemma A2.4.8 in [5] implies that the poset of subobjects X is isomorphic to the poset of subobjects of the terminal object of $\mathcal{E}/X$. So we need to prove that the terminal object of $\mathcal{E}/X$ is an atom. Now, as coproducts in $\mathcal{E}/X$ are calculated as in $\mathcal{E}$, the terminal object of $\mathcal{E}/X$ is indecomposable. Moreover, as $\mathcal{E}_{\text{ex}}$ is a topos then so is $(\mathcal{E}/X)_{\text{ex}} \cong \mathcal{E}_{\text{ex}}/X$.

Assume that the terminal object of $\mathcal{E}/X$ is not an atom. Then it has a strict and non-initial subobject. The induced geometric morphism $\mathcal{E}/X \to \widehat{\mathbf{2}}$ is surjective by Proposition 4.3 and helpful by Proposition 5.4. But this is absurd by Lemma 2.4. □

6. The case of toposes with enough points

In this section we prove Theorem 1.5. Again, this will involve proving that certain geometric morphisms $\mathcal{F} \to \widehat{\mathbf{2}}$ are helpful, but in the present case, it is convenient to restrict attention to the helping of maps with indecomposable codomain.

Lemma 6.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be extensive categories with finite limits and let $L \dashv R : \mathcal{C} \to \mathcal{D}$ be an adjunction. If the adjunction $L \dashv R$ helps every map in the family $\{a_i : A_i \to B_i\}_{i\in I}$ then $L \dashv R$ helps the map $\sum_i a_i : \sum_i A_i \to \sum_i B_i$.*

Proof. Using that the following diagram commutes:

$$\begin{array}{ccc}
\sum_{i\in I} B_i & \xrightarrow{\quad \eta \quad} & RL\sum_{i\in I} B_i \\
{\scriptstyle \sum_{i\in I}\eta_{B_i}}\downarrow & & \uparrow{\scriptstyle \cong\ R[\ldots,Lin_i,\ldots]} \\
\sum_{i\in I} RLB_i & \xrightarrow[{[\ldots,Rin_i,\ldots]}]{} & R(\sum_{i\in I} LB_i)
\end{array}$$

together with extensivity one proves that the pullback of $RL\sum_i a_i$ along η is isomorphic to the coproduct of the pullbacks of the (RLa_i)'s along the η_{B_i}'s. The rest is straightforward. □

In particular, for a geometric morphism with locally connected codomain to be helpful it is enough to check that the morphism helps maps with indecomposable codomain. This is particularly useful for geometric morphisms $\mathcal{F} \to \widehat{\mathbf{2}}$ for which the unit $\eta : C_I \to f_* f^* C_I$ is an iso for each set I. Such geometric morphisms arise in our present situation.

For the rest of the section let X be a sober topological space and let x in X be such that $\{x\}$ is closed. Moreover, let $f : \mathrm{Sh}(X) \to \widehat{\mathbf{2}}$ classify the complement of $\{x\}$.

Lemma 6.2. *The unit $\eta : C_I \to f_* f^* C_I$ is an iso for each I.*

Proof. By Lemma 4.2 we already know that $f_* f^* C_I$ is indecomposable so we need only calculate $(f_* f^* C_I)\top$. The representation of C_I as a space over the Sierpinski space Σ has a singleton as the fiber over the open point in Σ and I as the fiber over the closed point. It is then clear that, as a space over X, $f^* C_I$ has I as the fiber over x and singletons as fibers over the rest of the points of X. So there exist exactly I sections of $f^* C_I \to X$ (each of which assigns an element of I to x). □

Let us reformulate the two previous results as follows.

Lemma 6.3. *The geometric morphism f is helpful if and only if for every indecomposable Y in $\widehat{\mathbf{2}}$ and map $a : A \to Y$, the morphism $\eta^{-1}(f_*(f^* a)) : f_* f^* A \to f_* f^* Y \to Y$ factors through a.*

Let J be an index set and $\{I_j\}_{j\in J}$ a collection of sets. We now want to give a simple explicit description of $f_* f^*(\sum_{j\in J} C_{I_j}) \cong f_* \sum_{j\in J} f^* C_{I_j}$. In order to do this we define a *pointed disjoint covering of* X to be a pair $(\{V_j\}_{j\in J}, v)$ where the first component is a disjoint covering of X and v is an element of I_k for k uniquely determined by the condition $x \in V_k$.

Lemma 6.4. *The set $f_*(f^* \sum_{j\in J} C_{I_j})\top$ can be described as the set of pointed disjoint coverings of X.*

Proof. The global sections of the local homeo $\sum_{j\in J} f^*C_{I_j} \to X$ correspond to disjoint coverings $\{V_j\}_{j\in J}$ of X together with a family of cross sections $\{s_j : V_j \to f^*C_{I_j}\}_{j\in J}$. If we let k be determined by the condition $x \in V_k$ as above then it is clear that for $l \neq k$, there is only one cross section $V_l \to f^*C_{I_l}$, because the fibers over the elements that are not x are singletons. For k, the cross section $V_k \to f^*C_{I_k}$ is simply a choice of an element in I_k. □

An analogous argument shows that $f_*(f^* \sum_{j\in J} C_{I_j})\bot$ is simply the set of disjoint coverings $\{V_j\}_{j\in J}$ of the complement of $\{x\}$.

Proposition 6.5. *Let X be a sober topological space and let $x \in X$ be such that $\{x\}$ is closed. Moreover, let U be the complement of $\{x\}$ and let $f : \mathrm{Sh}(X) \to \widehat{\mathbf{2}}$ be its classifying map. If $\{x\}$ is not open then f is helpful.*

Proof. We use the sufficient condition stated in Lemma 6.3. So let $\{a_j : C_{I_j} \to Y\}_{j\in J}$ be a J-indexed collection of maps with indecomposable domain and codomain. Using the description of $P = f_* \sum_J f^*C_{I_j}$ discussed in Lemma 6.4 it is easy to conclude that the projection $\pi : P \to Y$ assigns to a pointed disjoint covering $(\{V_j\}_{j\in J}, v)$ the element $a_k v$ (recall that k is determined by the condition $x \in V_k$).

We now define a map $\gamma : P \to \sum_{j\in J} C_{I_j}$ over Y, that is, such that $a\gamma = \pi$. Let us start with $\gamma_\top$. The condition $a\gamma = \pi$ forces us to define $\gamma(\{V_j\}_{j\in J}, v) = v \in I_k = C_{I_k}\top$. So let us concentrate on $\gamma_\bot$. The elements of $P\bot$ are simply given by disjoint coverings of U. By projectivity, we need only define γ for those coverings that are restrictions of elements in $P\top$. Naturality would then determine the definition of $\gamma_\bot$ if each disjoint covering of U could be extended in at most one way to a disjoint covering of X. This is where the hypothesis that $\{x\}$ be not open comes in: let $\{V_j\}_{j\in J}$ be a disjoint covering of U and suppose that $V_k \cup \{x\}$ and $V_l \cup \{x\}$ are open. Then their intersection $(V_k \cap V_l) \cup \{x\}$ is open. As $\{x\}$ is not open, $k = l$. □

To prove Theorem 1.5 we need to show that $\mathrm{Sh}(X)_{\mathrm{ex}}$ a topos implies X discrete.

Proof of Theorem 1.5. We show that for every $x \in X$, $\{x\}$ is open. By Lemma 2.7 we can assume that X is T_1. So $\{x\}$ is closed. Then, its complement in X induces a subobject of 1 in $\mathrm{Sh}(X)$. Let $f : \mathrm{Sh}(X) \to \widehat{\mathbf{2}}$ be the classifying morphism of this subobject. If $\{x\}$ is not open then f is helpful by Proposition 6.5. So $\mathrm{Sh}(X)_{\mathrm{ex}}$ cannot be a topos. □

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